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NEW YORK UNIVERSITY
Courant Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. EM-189

Diffraction of a High-Frequency Plane Electromagnetic Wave by a Perfectly Conducting Circular Disc

D. S. JONES

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Abstract

A method already developed for the diffraction of scalar waves by a circular disc is generalised so as to be applicable to the diffraction of an electromagnetic wave. The generalisation leads, in a straightforward manner, to an asymptotic development of the current density on the disc at high frequencies. The principal complication which arises is in solving three simultaneous linear equations for three constants which occur in the method of solution. Detailed calculations are given for a normally incident plane wave.

Introduction

The problem of the diffraction of an electromagnetic wave by a circular disc has been the subject of much investigation. A solution was published by Möglich [1] as early as 1927. However, the solution which satisfies the edge conditions was not obtained until 15 years ago. This solution, which is expressed in terms of spheroidal functions, has been derived by Meixner [2], Meixner and Andrejewski [3], Flammer [4], and Belkina [5] when the primary field is a plane wave. The corresponding solution for a dipole field has been found by Meixner [6], Flammer [7] and Belkina [5]. Other exact analyses have been given by Nomura and Katsura [8] and Lur'e [9]. Numerical results up to ka = 10, k being the wave number and a the radius, have been obtained by Andrejewski [10,11].

Several approximate methods for tackling the problem have been devised, some are appropriate to low frequencies and some to high frequencies. At high frequencies the series of spheroidal functions become unmanageable so that other approaches have to be used. One method, based on physical reasoning, is due to Clemmow [12] and Millar [13,14] - this gives the dominant part of the field. The method of Seshradi and Wu [15] is based on an idea of Levine and Wu [16], which chooses the dominant part of the kernel of an integral equation so that it can be solved by the Wiener-Hopf technique.

The object of the following paper is to indicate a systematic process which provides as many terms as one is prepared to calculate of the asymptotic development at high frequencies. It is based on a theory which has already been successfully applied to the diffraction of scalar waves (Jones [17]). The generalisation required is fairly straightforward because the vector integral equation of electromagnetic waves can be reduced to relatively simple scalar integral equations (Jones [18]).

In Section 1 the salient points of this reduction are repeated. Section 2 is concerned with converting a typical integral equation into a singular integral equation of the first kind which is especially suitable for handling at high frequencies. From that point onwards the analysis is specialised to a normally incident plane wave. Only two integral equations have then to be solved. The asymptotic solution of one is derived in Section 3 and of the other in Section 4. The solutions, so obtained, involve three constants which have to be chosen so that the field behaves in a certain way near the edge. These constants are found in Section 5. A general formula for the scattering coefficient is derived in Section 6 and its evaluation for high frequencies is given in Section 7.

In order not to interrupt the argument of the main text, various calculations have been carried out in appendices. References to equations in appendices are written, for example, (A1), meaning equation (1) of Appendix A.

1. The integral equation

Let all lengths be normalised so that the radius of the disc is unity and choose coordinates so that the disc occupies $x^2 + y^2 \leq 1$, $z = 0$. Let the incident plane wave have electric vector $\underline{E}_o$ with Cartesian components given by

$$\underline{E}_o = (\cos \varphi_o \cos \theta_o, \sin \varphi_o, - \cos \varphi_o \sin \theta_o)e^{-i\alpha(x \sin \theta_o + z \cos \theta_o)},$$

the time dependence $e^{i\omega t}$ being understood and omitted throughout. The plane wave is propagating in a direction whose polar angles are θ_o, φ_o; in the case of normal incidence $\theta_o = 0$ and $\varphi_o = 0$. The constant α is the product of the wave number k and radius a of the disc; it will be taken as a positive real number throughout.

The total electric field at the point $\underline{R}$ can be written in the form

$$(1) \qquad \underline{E}(\underline{R}) = \underline{E}_o(\underline{R}) + (\text{grad div}_2 + \alpha^2) \int_S \underline{J}(\underline{R}_1)\, \psi(\underline{R},\underline{R}_1) dS$$

where S is the unit circle on $z = 0$, $\underline{R}_1$ is a point of S, $\psi(\underline{R},\underline{R}_1) = e^{-i\alpha|\underline{R}-\underline{R}_1|}/|\underline{R}-\underline{R}_1|$, div_2 is the two-dimensional operator independent of z and $\underline{J}$ is a constant multiple of the current density induced on the disc. Since the disc is perfectly conducting the tangential components of $\underline{E}$ vanish on the disc and so the integral equation

$$(2) \qquad \underline{E}_{ot} + (\mathrm{grad}_2 \mathrm{div}_2 + \alpha^2) \int_S \underline{J}(\underline{R}_1)\psi(\underline{R},\underline{R}_1)dS = 0 \qquad (\underline{R} \text{ in } S)$$

is obtained to determine $\underline{J}$.

The integral equation (2) can be converted into a set of integral equations for the Fourier components of the current and charge densities (Jones [18]). Recently, other approaches have been given by Lur'e [9] and Williams [19]. Let $\rho = \mathrm{div}_2\, \underline{J}$ and let r,φ be polar coordinates on the disc. Since ρ and $\underline{J}$ are periodic, with period 2π, in φ we can write

$$(3) \qquad J_r + iJ_\varphi = \sum_{n=-\infty}^{\infty} a_n(r)e^{in\varphi}, \quad J_r - iJ_\varphi = \sum_{n=-\infty}^{\infty} b_n(r)e^{in\varphi}, \quad \rho = \sum_{n=-\infty}^{\infty} \rho_n(r)e^{in\varphi}.$$

Then it has been shown (Jones [18]) that ρ_n, a_n and b_n satisfy, for $0 \leq r \leq 1$,

$$(4) \qquad -i \tan \Theta_o \cos \varphi_o e^{-\frac{1}{2}n\pi i} J_n(\alpha \sin \Theta_o) + A_n J_n(\alpha r) + \alpha \int_0^1 \rho_n(r_1)\psi_n(r,r_1)r_1 dr_1 = 0,$$

$$(5) \qquad \frac{1}{2\pi}\int_0^{2\pi} (E_{or} + iE_{o\varphi})e^{-in\varphi}d\varphi + \left(\frac{d}{dr} - \frac{n}{r}\right) \int_0^1 \rho_n(r_1)\psi_n(r,r_1)r_1\, dr_1 + \alpha^2 \int_0^1 a_n(r_1)\psi_{n+1}(r,r_1)r_1 dr_1 = 0,$$

$$(6) \qquad \frac{1}{2\pi}\int_0^{2\pi} (E_{or} - iE_{o\varphi})e^{-in\varphi}d\varphi + \left(\frac{d}{dr} + \frac{n}{r}\right) \int_0^1 \rho_n(r_1)\psi_n(r,r_1)r_1 dr_1 + \alpha^2 \int_0^1 b_n(r_1)\psi_{n-1}(r,r_1)r_1 dr_1 = 0$$

where A_n is an arbitrary constant to be determined eventually by the edge conditions and

$$(7) \qquad \psi_n(r,r_1) = \int_0^{2\pi} \frac{e^{-i\alpha(r^2+r_1^2-2rr_1\cos\varphi)^{\frac{1}{2}}}}{(r^2+r_1^2-2rr_1\cos\varphi)^{\frac{1}{2}}} \cos n\varphi \, d\varphi.$$

Substitution for ρ_n from (4) in (5) and (6) leads to the integral equations

$$(8) \quad e^{-\frac{1}{2}(n+1)\pi i}(\cos\varphi_o \sec\Theta_o + i\sin\varphi_o)J_{n+1}(\alpha\sin\Theta_o)$$
$$+ A_n J_{n+1}(\alpha r) + \alpha^2 \int_0^1 a_n(r_1)\psi_{n+1}(r,r_1)r_1 dr_1 = 0,$$

$$(9) \quad e^{-\frac{1}{2}(n-1)\pi i}(\cos\varphi_o \sec\Theta_o - i\sin\varphi_o)J_{n-1}(\alpha\sin\Theta_o)$$
$$- A_n J_{n-1}(\alpha r) + \alpha^2 \int_0^1 b_n(r_1)\psi_{n-1}(r,r_1)r_1 dr_1 = 0,$$

when the specified form of $\underline{E}_o$ is used in (5) and (6). In the particular case of normal incidence ($\Theta_o = 0$, $\varphi_o = 0$) the incident wave term disappears from (4), is absent from (8) unless $n = -1$ and from (9) unless $n = 1$. Since there will be no field if there is no incident wave we see that $a_n \equiv 0$, $b_n \equiv 0$ ($n \neq \pm 1$) and that

$$(10) \quad 1 + A_{-1}J_o(\alpha r) + \alpha^2 \int_0^1 a_{-1}(r_1)\psi_o(r,r_1)r_1 dr_1 = 0,$$

$$(11) \quad 1 - A_1 J_o(\alpha r) + \alpha^2 \int_0^1 b_1(r_1)\psi_o(r,r_1)r_1 dr_1 = 0.$$

Since A_{-1} and A_1 may not be zero we must also consider the equations for a_1 and b_{-1}, namely

$$A_1 J_2(\alpha r) + \alpha^2 \int_0^1 a_1(r_1)\psi_2(r,r_1)r_1 dr_1 = 0 , \tag{12}$$

$$-A_{-1} J_2(\alpha r) + \alpha^2 \int_0^1 b_{-1}(r_1)\psi_2(r,r_1)r_1 dr_1 = 0 , \tag{13}$$

since $\psi_{-2} = \psi_2$. Thus, only ρ_1 and ρ_{-1} will be non-zero; a fact which is also evident from

$$\rho_n = \frac{1}{2r} \frac{d}{dr} \{r(a_n + b_n)\} + \frac{n}{2r}(a_n - b_n).$$

At the edge the radial component of the current must vanish (Jones [20]) and this requires $a_1 + b_1 = 0$, $a_{-1} + b_{-1} = 0$ at $r = 1$. From (10)-(13) it is immediately clear that $A_1 = -A_{-1}$ and that $a_{-1} = b_1$, $a_1 = b_{-1}$. Hence

$$J_r = (a_1 + b_1)\cos\varphi, \quad J_\varphi = (a_1 - b_1)\sin\varphi \quad \text{and} \quad \rho = \frac{1}{r} \frac{d}{dr}(a_1 + b_1) + \frac{2a_1}{r} .$$

The further edge conditions that J_φ and ρ be integrable will be met if neither a_1 nor b_1 is more singular than $(1 - r)^{-\frac{1}{2}}$ as $r \to 1$.

2. The transformed integral equation.

All the integral equations (4), (8)-(13) are of the same type, namely

$$\int_0^1 r_1 k(r_1)\psi_n(r,r_1)dr_1 = g(r) \qquad (0 \le r \le 1) \tag{14}$$

where $g(r)$ is of specified form and $k(r_1)$ is to be found. To convert this integral equation to one of a more convenient type we use the result, proved in Appendix A, (A4)

$$\psi_n(r,r_1) = 2\pi(-)^n \frac{r^n}{\alpha^{n-1} r_1^n} \left(\frac{d}{rdr}\right)^n \left[\int_0^{\min(r,r_1)} J_{n-\frac{1}{2}}\{\alpha(r_1^2 - x^2)^{\frac{1}{2}}\} J_{-\frac{1}{2}}\{\alpha(r^2 - x^2)^{\frac{1}{2}}\} \times \frac{(r_1^2 - x^2)^{\frac{1}{2}(n-\frac{1}{2})}}{(r^2 - x^2)^{1/4}}\, dx \right.$$

$$\left. - i \int_0^{r_1} J_{n-\frac{1}{2}}\{\alpha(r_1^2 - x^2)^{\frac{1}{2}}\} J_{\frac{1}{2}}\{\alpha(r^2 - x^2)^{\frac{1}{2}}\} \frac{(r_1^2 - x^2)^{\frac{1}{2}(n-\frac{1}{2})}}{(r^2 - x^2)^{1/4}}\, dx \right].$$

Thus (14) can be written in the form

$$(15) \quad 2\pi(-)^n \frac{r^n}{\alpha^{n-1}} \left(\frac{d}{rdr}\right)^n \left[\int_0^r \frac{J_{-\frac{1}{2}}\{\alpha(r^2 - x^2)^{\frac{1}{2}}\}}{(r^2 - x^2)^{1/4}}\, h(x)dx - i \int_0^1 \frac{J_{\frac{1}{2}}\{\alpha(r^2 - x^2)^{\frac{1}{2}}\}}{(r^2 - x^2)^{1/4}}\, h(x)dx \right] = g(r)$$

in $0 \leq r \leq 1$ where

$$(16) \quad h(x) = \int_x^1 J_{n-\frac{1}{2}}\{\alpha(r_1^2 - x^2)^{\frac{1}{2}}\}(r_1^2 - x^2)^{\frac{1}{2}(n-\frac{1}{2})} k(r_1) r_1^{1-n} dr_1 .$$

There are two methods of tackling (15). One, which leads to equations similar to those obtained by the author earlier (Jones [17]), will be given here. The other, which leads to equations analogous to those of Bazer and Hochstadt [21], is described in Appendix B.

Let

$$\kappa(r_1) = \int_{r_1}^{1} \frac{(u^2 - r_1^2)^{n-1}}{(n-1)!\,2^{n-1}} \, \frac{k(u)}{u^{n-1}} \, du \tag{17}$$

for $n \geq 1$ and $\kappa(r_1) = k(r_1)$ for $n = 0$. Then

$$\left(\frac{d}{r_1 dr_1}\right)^n \kappa(r_1) = \left(\frac{d}{r_1 dr_1}\right)^{n-1} \int_{r_1}^{1} \frac{(-)(u^2 - r_1^2)^{n-2}}{(n-2)!\;2^{n-2}} \, \frac{k(u)}{u^{n-1}} \, du$$

$$= \frac{d}{r_1 dr_1} \int_{r_1}^{1} (-)^{n-1} \frac{k(u)}{u^{n-1}} \, du$$

$$= (-)^n \, k(r_1)/r_1^n \, . \tag{18}$$

Note that because $k(u)$ is no more singular than $(1-u^2)^{-\frac{1}{2}}$ as $u \to 1$, (17) shows that $\kappa(r_1)$ is proportional to $(1-r_1^2)^{n-\frac{1}{2}}$ as $r_1 \to 1$. Further, since $k(u)$ is bounded at $u = 0$, $\kappa(r_1)$ is bounded at $r_1 = 0$.

On account of (18), (16) can be written

$$h(x) = \int_{x}^{1} J_{n-\frac{1}{2}}\{\alpha(r_1^2 - x^2)^{\frac{1}{2}}\} \, (r_1^2 - x^2)^{\frac{1}{2}(n-\frac{1}{2})} r_1 (-)^n \left(\frac{d}{r_1 dr_1}\right)^n \kappa(r_1) dr_1 .$$

For $n = 0$ this formula need not be modified. For $n \geq 1$, integrate by parts in such a way as to reduce the derivative on κ. Because $\frac{1}{z}\frac{d}{dz}\{z^\nu J_\nu(z)\} = z^{\nu-1} J_{\nu-1}(z)$ we have

$$h(x) = \left[J_{n-\frac{1}{2}}\{\alpha(r_1^2 - x^2)^{\frac{1}{2}}\}\, (r_1^2 - x^2)^{\frac{1}{2}(n-\frac{1}{2})} (-)^n \left(\frac{d}{r_1 dr_1}\right)^{n-1} \kappa(r_1) \right]_x^1$$

$$- \alpha \int_x^1 J_{n-\frac{3}{2}}\{\alpha(r_1 - x^2)^{\frac{1}{2}}\}(r_1^2 - x^2)^{\frac{1}{2}(n-\frac{3}{2})} r_1 (-)^n \left(\frac{d}{r_1 dr_1}\right)^{n-1} \kappa(r_1) dr_1$$

$$= \alpha \int_x^1 J_{n-\frac{3}{2}}\{\alpha(r_1^2 - x^2)^{\frac{1}{2}}\}(r_1^2 - x^2)^{\frac{1}{2}(n-\frac{3}{2})} r_1 (-)^{n-1} \left(\frac{d}{r_1 dr_1}\right)^{n-1} \kappa(r_1) dr_1$$

since the first term vanishes because of the behavior of the Bessel function near $r_1 = x$ and of κ near $r_1 = 1$. The remaining term is a constant multiple of the original integral but with $n - 1$ for n. Repetition of the process leads to

$$h(x) = \alpha^n \int_x^1 \frac{J_{-\frac{1}{2}}\{\alpha(r_1^2 - x^2)^{\frac{1}{2}}\}}{(r_1^2 - x^2)^{\frac{1}{4}}} r_1 \kappa(r_1) dr_1 . \tag{19}$$

This is also valid for $n = 0$ in view of the remark above.

Substitute (19) in (15) and interchange the order of integration in the second term on the left-hand side. The result is

$$\begin{aligned} 2\pi(-)^n r^n \alpha \left(\frac{d}{rdr}\right)^n \Bigg[& \int_0^r \frac{J_{-\frac{1}{2}}\{\alpha(r^2 - x^2)^{\frac{1}{2}}\}}{(r^2 - x^2)^{1/4}} \int_x^1 \frac{J_{-\frac{1}{2}}\{\alpha(r_1^2 - x^2)^{\frac{1}{2}}\}}{(r_1^2 - x^2)^{1/4}} r_1 \kappa(r_1) dr_1 \\ & - i \int_0^1 r_1 \kappa(r_1) \int_0^{r_1} \frac{J_{-\frac{1}{2}}\{\alpha(r_1^2 - x^2)^{\frac{1}{2}}\}}{(r_1^2 - x^2)^{1/4}} \frac{J_{\frac{1}{2}}\{\alpha(r^2 - x^2)^{\frac{1}{2}}\}}{(r^2 - x^2)^{1/4}} dx\, dr_1 \Bigg] = g(r). \end{aligned} \tag{20}$$

But it has been shown elsewhere (Jones [17]) that

$$\int_0^{r_1} \frac{J_{-\frac{1}{2}}\{\alpha(r_1^2 - x^2)^{\frac{1}{2}}\}}{(r_1^2 - x^2)^{1/4}} \, \frac{J_{\frac{1}{2}}\{\alpha(r^2 - x^2)^{\frac{1}{2}}\}}{(r^2 - x^2)^{1/4}} \, dx = \int_0^{r} \frac{J_{-\frac{1}{2}}\{\alpha(r^2 - x^2)^{\frac{1}{2}}\}}{(r^2 - x^2)^{1/4}} \, \frac{J_{\frac{1}{2}}\{\alpha(r_1^2 - x^2)^{\frac{1}{2}}\}}{(r_1^2 - x^2)^{1/4}} \, dx.$$

Therefore (20) can be written as

$$(21) \qquad 2\pi(-)^n r^n \alpha \left(\frac{d}{rdr}\right)^n \int_0^r \frac{J_{-\frac{1}{2}}\{\alpha(r^2 - x^2)^{\frac{1}{2}}\}}{(r^2 - x^2)^{1/4}} \lambda(x)dx = g(r)$$

where

$$(22) \; \lambda(x) = \int_x^1 \frac{J_{-\frac{1}{2}}\{\alpha(r_1^2 - x^2)^{\frac{1}{2}}\}}{(r_1^2 - x^2)^{1/4}} r_1 \kappa(r_1)dr_1 - i \int_0^1 \frac{J_{\frac{1}{2}}\{\alpha(r_1^2 - x^2)^{\frac{1}{2}}\}}{(r_1^2 - x^2)^{1/4}} r_1 \kappa(r_1)dr_1 .$$

Integration of (21) n times gives

$$\int_0^r \frac{J_{-\frac{1}{2}}\{\alpha(r^2 - x^2)^{\frac{1}{2}}\}}{(r^2 - x^2)^{1/4}} \lambda(x)dx = \sum_{m=0}^{n-1} c_m r^{2m} - \frac{1}{(n-1)!\,2^n \pi\alpha} \int_0^r (u^2 - r^2)^{n-1} \frac{g(u)}{u^{n-1}} du$$

where $c_o, c_1, \ldots, c_{n-1}$ are arbitrary constants. If $n = 0$ the right-hand side is replaced by $g(r)/2\pi\alpha$.

Replace the Bessel function by its trigonometric equivalent and then

$$\int_0^r \frac{\cos\{\alpha(r^2 - x^2)^{\frac{1}{2}}\}}{(r^2 - x^2)^{\frac{1}{2}}} \lambda(x)dx = \sum_{m=0}^{n-1} c'_m r^{2m} - \frac{1}{(n-1)!\,2^{n+\frac{1}{2}} \pi^{\frac{1}{2}} \alpha^{\frac{1}{2}}} \int_0^r (u^2 - r^2)^{n-1} \frac{g(u)}{u^{n-1}} du .$$

If $n = 0$ the right hand side is $g(r)/2^{3/2}\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}$.

The solution of this integral equation is known (Jones [22]) to be

$$(23)\quad \lambda(x) = \frac{1}{\pi}\frac{d}{dx}\int_0^{x^2}\frac{\cosh\{\alpha(x^2-y)^{\frac{1}{2}}\}}{(x^2-y)^{\frac{1}{2}}}\left[\sum_{m=0}^{n-1} c'_m y^m - \frac{1}{(n-1)!\,2^{n+\frac{3}{2}}\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}}\int_0^y (u-y)^{n-1}\frac{g(u^{\frac{1}{2}})}{u^{\frac{1}{2}n}}\,du\right]dy\,.$$

If $n = 0$ the term in [ ] is replaced by $g(y^{\frac{1}{2}})/2^{\frac{3}{2}}\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}$.

Now that $\lambda(x)$ is known $\kappa(r_1)$ is determined from (22), which may be expressed as

$$\int_x^1 \frac{\cos\{\alpha(r_1^2-x^2)^{\frac{1}{2}}\}}{(r_1^2-x^2)^{\frac{1}{2}}}\, r_1\kappa(r_1)dr_1 - i\int_0^1 \frac{\sin\{\alpha(r_1^2-x^2)^{\frac{1}{2}}\}}{(r_1^2-x^2)^{\frac{1}{2}}}\, r_1\kappa(r_1)dr_1 = (\tfrac{1}{2}\pi\alpha)^{\frac{1}{2}}\lambda(x).$$

Take as new variables $z = x^2$, $t = r_1^2$ and the equation becomes

$$(24)\quad \int_z^1 \frac{\cos\{\alpha(t-z)^{\frac{1}{2}}\}}{(t-z)^{\frac{1}{2}}}\,\kappa(t^{\frac{1}{2}})dt - i\int_0^1 \frac{\sin\{\alpha(t-z)^{\frac{1}{2}}\}}{(t-z)^{\frac{1}{2}}}\,\kappa(t^{\frac{1}{2}})dt = (2\pi\alpha)^{\frac{1}{2}}\lambda(z^{\frac{1}{2}})$$

valid for $0 < z < 1$. In this form the integral equation is of the same type as (6) in our earlier paper (Jones [17]). Hence we deduce that

$$(25)\quad \int_0^1 w\kappa(w)\left\{\frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v}\right\}dw = K^{(0)}(v) \qquad (0 < v < 1)$$

where

$$(26)\quad K^{(0)}(v) = (\tfrac{1}{2}\pi\alpha)^{\frac{1}{2}}\frac{d}{dv}\int_0^{v^2}\frac{\lambda(z^{\frac{1}{2}})\cos\{\alpha(v^2-z)^{\frac{1}{2}}\}}{(v^2-z)^{\frac{1}{2}}}\,dz$$

and the principal value of the integral in (25) is to be taken. Subsequently, a principal value will always be assumed where necessary.

A solution of (25) is required that has no more singularities than the solution of the corresponding integral equation in our earlier paper so that the same method as was used there may be adopted here to find an asymptotic development for large α. This asymptotic development will involve the constants c'_m because of (23) and these constants will have to be chosen so that $\kappa(r_1)$ behaves like $(1 - r_1^2)^{n-\frac{1}{2}}$ as $r_1 \to 1$. Once κ is known $k(r_1)$ can be determined from (18). Thus the distribution on the disc will be known and all other quantities can be deduced therefrom.

In order to illustrate how the details work out in practice we shall now consider the particular case of the plane wave at normal incidence.

3. The plane wave at normal incidence.

It has been pointed out in Section 1 that, for the plane wave at normal incidence, it is necessary to determine only a_1 and b_1. In this section we consider the determination of b_1. Equation (11) is of the same form as (14) with $n = 0$, $k = b_1$ and $g(r) = \{A_1 J_o(\alpha r) - 1\}/\alpha^2$. Hence the theory of the preceding section shows that

$$(27) \quad \int_o^1 w b_1(w) \left\{ \frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v} \right\} dw = B^{(o)}(v)$$

where

$$B^{(o)}(v) = (\tfrac{1}{2}\pi\alpha)^{\frac{1}{2}} \frac{d}{dv} \int_o^{v^2} \frac{\cos\{\alpha(v^2-z)^{\frac{1}{2}}\}}{(v^2-z)^{\frac{1}{2}}} . \frac{2}{\pi} z^{\frac{1}{2}} \frac{d}{dz} \int_o^z \frac{\cosh\{\alpha(z-y)^{\frac{1}{2}}\}}{(z-y)^{\frac{1}{2}}} \left[\frac{A_1 J_o(\alpha y^{\frac{1}{2}}) - 1}{2^{3/2}\pi^{\frac{1}{2}}\alpha^{5/2}} \right] dy \, dz.$$

Now

$$\int_0^z J_o(\alpha y^{\frac{1}{2}}) \frac{\cosh\{\alpha(z-y)^{\frac{1}{2}}\}}{(z-y)^{\frac{1}{2}}}\, dy = \lim_{\alpha_1 \to \alpha} \int_0^z J_o(\alpha_1 y^{\frac{1}{2}}) \frac{\cosh\{\alpha(z-y)^{\frac{1}{2}}\}}{(z-y)^{\frac{1}{2}}}\, dy$$

$$= \lim_{\alpha_1 \to \alpha} \int_0^z J_o(\alpha_1 y^{\frac{1}{2}})(\tfrac{1}{2}\pi\alpha)^{\frac{1}{2}} \frac{I_{-\frac{1}{2}}\{\alpha(z-y)^{\frac{1}{2}}\}}{(z-y)^{1/4}}\, dy$$

$$= \lim_{\alpha_1 \to \alpha} \int_0^{\frac{1}{2}\pi} J_o(\alpha_1 z^{\frac{1}{2}}\sin\,\Theta)(2\pi\alpha)^{\frac{1}{2}} I_{-\frac{1}{2}}(\alpha z^{\frac{1}{2}}\cos\,\Theta) z^{\frac{3}{4}} \sin\Theta \cos^{\frac{1}{2}}\Theta d\Theta$$

on putting $y = z\sin^2\Theta$. This last integral is a particular case of Sonine's second finite integral (Watson [24]) and so

$$\int_0^z J_o(\alpha y^{\frac{1}{2}}) \frac{\cosh\{\alpha(z-y)^{\frac{1}{2}}\}}{(z-y)^{\frac{1}{2}}}\, dy = \lim_{\alpha_1 \to \alpha} (2\pi\alpha)^{\frac{1}{2}} z^{\frac{3}{4}} (\alpha z^{\frac{1}{2}})^{-\frac{1}{2}} \frac{J_{\frac{1}{2}}\{z^{\frac{1}{2}}(\alpha_1^2-\alpha^2)^{\frac{1}{2}}\}}{z^{\frac{1}{4}}(\alpha_1^2-\alpha^2)^{\frac{1}{4}}}$$

$$= \lim_{\alpha_1 \to \alpha} 2 \frac{\sin\, z^{\frac{1}{2}}(\alpha_1^2-\alpha^2)^{\frac{1}{2}}}{(\alpha_1^2 - \alpha^2)^{\frac{1}{2}}}$$

$$= 2z^{\frac{1}{2}}\,. \qquad (28)$$

Hence

$$B^{(o)}(v) = \frac{1}{2\pi\alpha^2} \frac{d}{dv} \int_0^{v^2} \frac{\cos\{\alpha(v^2-z)^{\frac{1}{2}}\}}{(v^2-z)^{\frac{1}{2}}} (A_1 - \cosh \alpha z^{\frac{1}{2}}) dz$$

$$= \frac{1}{4\pi\alpha^2} \{4A_1 \cos\alpha v - 4\cos\alpha v - 4\alpha v\ \mathrm{Si}(\alpha v)\}$$

where $\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t} dt.$

The last two terms in the { } of $B^{(o)}$ are the same, apart from sign, as those which arise for the scattering of a scalar plane wave by a sound-soft disc. The part of b_1 due to these terms may therefore be deduced from our earlier paper. Denote by $f(w)$ the solution of (27) when $B^{(o)}(v) = 4 \cos \alpha v + 4 \alpha v \,\mathrm{Si}(\alpha v)$. Then

$$b_1(w) = \frac{1}{4\pi\alpha^2} \{A_1 d(w) - f(w)\} \tag{29}$$

where

$$\int_0^1 w d(w) \left\{ \frac{e^{-i\alpha(w+v)}}{w + v} + \frac{e^{-i\alpha(w-v)}}{w - v} \right\} dw = 4 \cos \alpha v. \tag{30}$$

Let

$$D(v) = \int_0^1 \frac{wd(w)}{w - v} e^{-i\alpha(w-v)} dw \; ; \tag{31}$$

then

$$d(w) = \frac{- e^{i\alpha w}}{\pi^2 w^{\frac{1}{2}} (1-w)^{\frac{1}{2}}} \int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \frac{D(v) e^{-i\alpha v}}{v - w} dv \tag{32}$$

as in our earlier paper. The equation satisfied by $D(v)$ is

$$D(v) = 4 \cos \alpha v - \left(\frac{v}{1+v}\right)^{\frac{1}{2}} \frac{e^{-i\alpha v}}{\pi} \int_0^1 \frac{D(t) e^{-i\alpha t}}{t + v} \left(\frac{1-t}{t}\right)^{\frac{1}{2}} dt$$

for v in $(0,1)$. Write

$$D(v) = 2e^{i\alpha v} + \left(\frac{v}{1+v}\right)^{\frac{1}{2}} e^{-i\alpha v} G(v)$$

and then

$$G(v) = 2 - \frac{1}{\pi} \int_0^1 \left(\frac{1-t}{1+t}\right)^{\frac{1}{2}} \frac{G(t)e^{-2i\alpha t}}{t+v} dt .$$

This equation is analogous to equation (31) of our earlier paper. There is the difference that 2 replaces $h_1(v)$. However, the argument which followed (31) can still be carried through. Hence we can form the iteration scheme

$$G_n(v) = \left(\frac{1+v}{v}\right)^{\frac{1}{2}} \int_0^\infty \frac{E_n(t)}{t-v} e^{-2i\alpha(t-v)} dt,$$

$$E_n(y) = \frac{i\alpha e^{i\alpha y}}{4\pi} \int_0^\infty \left(\frac{x}{1+x}\right)^{\frac{1}{2}} e_n(x)M(x,y)dx \qquad (n \geq 1),$$

$$e_n(v) = \int_1^\infty \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \frac{E_{n-1}(t)e^{-2i\alpha t}}{t+v} dt \qquad (n \geq 2),$$

(33) $$e_1(v) = 2$$

and

$$M(x,y) = \frac{4xyH_1^{(2)}(\alpha x)J_0(\alpha y) - 4y^2H_0^{(2)}(\alpha x)J_1(\alpha y)}{x^2 - y^2} .$$

From the general theory of our earlier paper we can say that each iteration is smaller than the preceding by the factor α^{-1}, as far as α is concerned. Also, from (32), the asymptotic development of $d(w)$ is given by

$$d(w) \sim - \frac{e^{i\alpha w}}{\pi^2 w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \frac{2dv}{v-w}$$

$$(34) \qquad + \sum_{n=1} \left\{ \frac{E_n(w)e^{-i\alpha w}}{w} + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_1^\infty \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \frac{E_n(t)e^{-2i\alpha t}}{t-w} dt \right\}$$

$$\sim \frac{2e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} + \sum_{n=1} \left\{ \frac{E_n(w)e^{-i\alpha w}}{w} + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_1^\infty \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \frac{E_n(t)e^{-2i\alpha t}}{t-w} dt \right\} .$$

The substitution of (34) in (29) together with the known form of f now gives b_1 completely apart from the unknown constant A_1.

4. The determination of a_1.

The integral equation for a_1 is (12). This is the same as (14) with the identification $n = 2$, $k(r_1) = a_1(r_1)$ and $g(r) = -A_1 J_2(\alpha r)/\alpha^2$. According to the theory of Section 2, and (17) in particular, we introduce the function $p_1(r_1)$ defined by

$$(35) \qquad p_1(r_1) = \frac{1}{2} \int_{r_1}^1 \frac{u^2 - r_1^2}{u} a_1(u)du$$

so that, on account of (18),

$$(36) \qquad a_1(r_1) = r_1^2 \left(\frac{d}{r_1 dr_1}\right)^2 p_1(r_1) .$$

The function $p_1(r_1)$ must be bounded at the origin and behave like $(1-r_1^2)^{\frac{3}{2}}$ as $r_1 \to 1$.

The function corresponding to $\lambda(x)$ is, from (23),

$$\frac{1}{\pi}\frac{d}{dx}\int_0^{x^2} \frac{\cosh\{\alpha(x^2-y)^{\frac{1}{2}}\}}{(x^2-y)^{\frac{1}{2}}} \left[c_0' + c_1'y + \frac{A_1}{2^{\frac{7}{2}}\alpha^{\frac{5}{2}}\pi^{\frac{1}{2}}} \int_0^y \frac{u-y}{u} \; J_2(\alpha u^{\frac{1}{2}})du\right] .$$

Now

$$\left(\frac{d}{udu}\right)^2 J_0(\alpha u) = \alpha^2 \frac{J_2(\alpha u)}{u^2}$$

so that

$$\begin{aligned}
\int_0^y \frac{u-y}{u} J_2(\alpha u^{\frac{1}{2}})du &= 2\int_0^{y^{\frac{1}{2}}} \left(\frac{u^2-y}{u}\right) J_2(\alpha u)du \\
&= \frac{2}{\alpha^2} \int_0^{y^{\frac{1}{2}}} u(u^2-y) \left(\frac{d}{udu}\right)^2 J_0(\alpha u)du \\
&= \frac{2}{\alpha^2}\left[(u^2-y)\frac{d}{udu} J_0(\alpha u)\right]_0^{y^{\frac{1}{2}}} - \frac{4}{\alpha^2}\int_0^{y^{\frac{1}{2}}} \frac{d}{du} J_0(\alpha u)du \\
&= -y + \frac{4}{\alpha^2}\{1 - J_0(\alpha y^{\frac{1}{2}})\} .
\end{aligned}$$

Hence the function corresponding to $\lambda(x)$ is

$$\frac{1}{\pi}\frac{d}{dx}\int_0^{x^2} \frac{\cosh\{\alpha(x^2-y)^{\frac{1}{2}}\}}{(x^2-y)^{\frac{1}{2}}} \left\{d_0 + d_1(y - 2/\alpha^2) - \frac{A_1 J_0(\alpha y^{\frac{1}{2}})}{(8\alpha^9\pi)^{\frac{1}{2}}}\right\} dy$$

where

$$d_0 = c'_0 + 2d_1/\alpha^2 + A_1/(8\alpha^9\pi)^{\frac{1}{2}}, \quad d_1 = c'_1 - A_1/(128\alpha^5\pi)^{\frac{1}{2}} .$$

On account of (28) the function corresponding to $\lambda(x)$ is

$$\frac{1}{\pi}\left\{2(d_0 - \frac{2d_1}{\alpha^2}) \cosh \alpha x + \frac{4d_1}{\alpha} x \sinh \alpha x - \frac{A_1}{(2\pi\alpha^9)^{\frac{1}{2}}}\right\} .$$

Therefore, from (25) and (26),

$$(37) \quad \int_0^1 wp_1(w)\left\{\frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v}\right\} dw = P_1^{(0)}(v)$$

where

$$P_1^{(0)}(v) = (\frac{\alpha}{2\pi})^{\frac{1}{2}} \frac{d}{dv} \int_0^{v^2} \frac{\cos\{\alpha(v^2-z)^{\frac{1}{2}}\}}{(v^2-z)^{\frac{1}{2}}}$$

$$\times \left\{2(d_0 - \frac{2d_1}{\alpha^2}) \cosh \alpha z^{\frac{1}{2}} + \frac{4d_1}{\alpha} z^{\frac{1}{2}} \sinh \alpha z^{\frac{1}{2}} - \frac{A_1}{(2\pi\alpha^9)^{\frac{1}{2}}}\right\} dz .$$

Now

$$P_1^{(0)}(v) = (\frac{\alpha}{2\pi})^{\frac{1}{2}}(d_0 - \frac{2d_1}{\alpha^2})\{4 \cos \alpha v + 4\alpha v \operatorname{Si}(\alpha v)\} - (\frac{\alpha}{2\pi})^{\frac{1}{2}} \frac{2A_1 \cos \alpha v}{(2\pi\alpha^9)^{\frac{1}{2}}}$$

$$+ (\frac{\alpha}{2\pi})^{\frac{1}{2}} \frac{d}{dv}\left[-\frac{8d_1 z^{\frac{1}{2}}}{\alpha^2} \sinh \alpha z^{\frac{1}{2}} \sin\{\alpha(v^2-z)^{\frac{1}{2}}\}\right]_0^{v^2}$$

$$+ (\frac{\alpha}{2\pi})^{\frac{1}{2}} \frac{d}{dv} \int_0^{v^2} \frac{4d_1}{\alpha^2} \sin\{\alpha(v^2-z)^{\frac{1}{2}}\}\left\{\frac{\sinh \alpha z^{\frac{1}{2}}}{z^{\frac{1}{2}}} + \alpha \cosh \alpha z^{\frac{1}{2}}\right\} dz$$

$$= \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}}\left(d_o - \frac{2d_1}{\alpha^2}\right)\{4\cos \alpha v + 4\alpha v \operatorname{Si}(\alpha v)\} - \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} \frac{2A_1 \cos \alpha v}{(2\pi\alpha^9)^{\frac{1}{2}}}$$

$$+ \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} \int_0^{v^2} \frac{4d_1 v}{\alpha} \frac{\cos\{\alpha(v^2-z)^{\frac{1}{2}}\}}{(v^2-z)^{\frac{1}{2}}} \left(\frac{\sinh \alpha z^{\frac{1}{2}}}{z^{\frac{1}{2}}} + \alpha \cosh \alpha z^{\frac{1}{2}}\right) dz$$

$$= \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} \left[\left(d_o - \frac{2d_1}{\alpha^2}\right)\{4\cos \alpha v + 4\alpha v \operatorname{Si}(\alpha v)\} - \frac{2A_1 \cos \alpha v}{(2\pi\alpha^9)^{\frac{1}{2}}} + \frac{8d_1 v}{\alpha} \operatorname{Si}(\alpha v\right.$$

$$\left. + \frac{4d_1 v}{\alpha}\{\sin \alpha v + \alpha v \cos \alpha v + \alpha^2 v^2 \operatorname{Si}(\alpha v)\}\right]$$

$$= \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} \left[d_o\{4\cos \alpha v + 4\alpha v \operatorname{Si}(\alpha v)\} - \frac{2A_1 \cos \alpha v}{(2\pi\alpha^9)^{\frac{1}{2}}}\right.$$

$$\left. + \frac{4d_1}{\alpha^2}\{-2\cos \alpha v + \alpha v \sin \alpha v + \alpha^2 v^2 \cos \alpha v + \alpha^3 v^3 \operatorname{Si}(\alpha v)\}\right] .$$

Consequently, remembering (30), we see that

$$(38) \qquad p_1(w) = \left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} \left\{d_o f(w) - \frac{A_1 d(w)}{(8\pi\alpha^9)^{\frac{1}{2}}} + d_1 p(w)\right\}$$

where

$$\int_0^1 wp(w) \left\{\frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v}\right\} dw = P^{(o)}(v)$$

and

(39) $$P^{(o)}(v) = \frac{4}{\alpha^2}\{-2\cos \alpha v + \alpha v \sin \alpha v + \alpha^2 v^2 \cos \alpha v + \alpha^3 v^3 \mathrm{Si}(\alpha v)\}.$$

Let

$$\int_o^1 wp(w)\, \frac{e^{-i\alpha(w-v)}}{w - v} = P(v) .$$

Since p_1, f and d are all bounded at the origin so is p. Hence

(40) $$p(w) = - \frac{e^{i\alpha w}}{\pi^2 w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_o^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \frac{P(v)e^{-i\alpha v}}{v - w}\, dv$$

and the equation satisfied by P(v) is

$$P(v) = P^{(o)}(v) - \left(\frac{v}{1+v}\right)^{\frac{1}{2}} \frac{e^{-i\alpha v}}{\pi} \int_o^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{P(t)e^{-i\alpha t}}{t + v}\, dt .$$

Put

$$P(v) = 2e^{i\alpha v}(\alpha^2 v^2 - i\alpha v - 2)/\alpha^2 + 2i\alpha v^3 \int_{-2v}^{\infty} \frac{e^{-i\alpha(t+v)}}{t + v}\, dt + P_1(v)$$

and use the results

$$\int_o^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{2(\alpha^2 t^2 - i\alpha t - 2)}{\alpha^2(t + v)}\, dt = \frac{2}{\alpha^2}\int_o^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \left\{\alpha^2 t - \alpha(i + \alpha v) + \frac{\alpha^2 v^2 + i\alpha v - 2}{t + v}\right\} dt$$

(41) $$= \frac{2\pi}{\alpha^2}\left[\frac{1}{8}\alpha^2 - \frac{1}{2}\alpha(i + \alpha v) + (\alpha^2 v^2 + i\alpha v - 2)\ \left\{\left(\frac{1+v}{v}\right)^{\frac{1}{2}} - 1\right\}\right],$$

$$\int_o^1 \left(\frac{1-w}{w}\right)^{\frac{1}{2}} \frac{2i\alpha w^3}{w + v} \int_o^{\infty} \frac{e^{-i\alpha t}}{t - w} dt\, dw = -2i\alpha \int_o^{\infty} e^{-i\alpha t} \int_o^1 \left(\frac{1-w}{w}\right)^{\frac{1}{2}} \left\{w+t-v + \frac{v^3}{(v+t)(w+v)} + \frac{t^3}{(w-t)(t+v)}\right\} dw\, dt$$

$$= -2i\alpha \int_0^\infty e^{-i\alpha t}\left[\frac{1}{8}\pi + \frac{1}{2}\pi(t-v) + \frac{v^3\pi}{t+v}\left\{\left(\frac{1+v}{v}\right)^{\frac{1}{2}} - 1\right\} + \frac{t^3\pi}{t+v}\left\{\left(\frac{t-1}{t}\right)^{\frac{1}{2}} H(t-1) - 1\right\}\right]dt$$

from (E10) of Appendix E of Jones [17]. The last integral can be written

$$-2\pi i\alpha \int_0^1 e^{-i\alpha t}\{\tfrac{1}{8} + \tfrac{1}{2}(t-v) - (t^2 - tv + v^2)\}dt - 2\pi i\alpha v^3\left(\frac{1+v}{v}\right)^{\frac{1}{2}} \int_0^\infty \frac{e^{-i\alpha t}}{t+v}\,dt$$

$$(42) \quad - 2\pi i\alpha \int_1^\infty e^{-i\alpha t}\left\{\frac{1}{8} + \frac{1}{2}(t-v) - (t^2 - tv + v^2) + \frac{t^3}{t+v}\left(\frac{t-1}{t}\right)^{\frac{1}{2}}\right\}dt$$

$$= -2\pi\left(\frac{1}{8} - \frac{1}{2}v - v^2\right) + \frac{2\pi i}{\alpha}\left(\frac{1}{2} + v\right) - \frac{4\pi}{\alpha^2} - 2\pi i\alpha v^3\left(\frac{1+v}{v}\right)^{\frac{1}{2}} \int_0^\infty \frac{e^{-i\alpha t}}{t+v}\,dt$$

$$- 2\pi i\alpha e^{-\frac{3}{4}\pi i - i\alpha} \int_0^\infty \frac{t^{\frac{1}{2}}(1-it)^{5/2}}{1+v-it} e^{-\alpha t}dt$$

on evaluating the first integral and deforming the contour of the third integral into the line joining 1 to 1 - i ∞. Hence

$$\int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{P(t) - P_1(t)}{t+v} e^{-i\alpha t}dt = \frac{2\pi}{\alpha^2}(\alpha^2 v^2 + i\alpha v - 2)\left(\frac{1+v}{v}\right)^{\frac{1}{2}} - 2\pi i\alpha v^3\left(\frac{1+v}{v}\right)^{\frac{1}{2}}$$

$$\times \int_0^\infty \frac{e^{-i\alpha t}}{t+v}\,dt - 2\pi\alpha e^{-\frac{1}{4}\pi i - i\alpha} \int_0^\infty \frac{t^{\frac{1}{2}}(1-it)^{\frac{5}{2}}}{1+v-it} e^{-\alpha t}dt$$

so that $P_1(v)$ satisfies

$$P_1(v) = 2\alpha e^{-\frac{1}{4}\pi i - i\alpha - i\alpha v}\left(\frac{v}{1+v}\right)^{\frac{1}{2}} \int_0^\infty \frac{t^{\frac{1}{2}}(1-it)^{\frac{5}{2}}}{1+v-it} e^{-\alpha t}\,dt - \left(\frac{v}{1+v}\right)^{\frac{1}{2}} \frac{e^{-i\alpha v}}{\pi}$$

$$\times \int_0^1 \left(\frac{1-t}{t}\right)^{\frac{1}{2}} \frac{P_1(t)e^{-i\alpha t}}{t+v}\,dt \;.$$

On putting $P_1(v) = \left(\frac{v}{1+v}\right)^{\frac{1}{2}} e^{-i\alpha v} R(v)$ we obtain an equation analogous to (31) of Jones [17], namely

$$R(v) = 2\alpha e^{-\frac{1}{4}\pi i - i\alpha} \int_0^\infty \frac{t^{\frac{1}{2}}(1-it)^{\frac{5}{2}}}{1+v-it} e^{-\alpha t} dt - \frac{1}{\pi} \int_0^1 \left(\frac{1-t}{1+t}\right)^{\frac{1}{2}} \frac{R(t) e^{-2i\alpha t}}{t+v} dt .$$

Therefore we form the iteration scheme

$$R_n(v) = \left(\frac{1+v}{v}\right)^{\frac{1}{2}} \int_0^\infty \frac{S_n(t) e^{-2i\alpha(t-v)}}{t-v} dt ,$$

$$(43) \qquad S_n(y) = \frac{i\alpha e^{i\alpha y}}{4\pi} \int_0^\infty \left(\frac{x}{1+x}\right)^{\frac{1}{2}} s_n(x)\, M(x,y) dx \qquad (n \geq 1) ,$$

$$(44) \qquad s_n(v) = \int_1^\infty \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \frac{S_{n-1}(t) e^{-2i\alpha t}}{t+v} dt \qquad (n \geq 2) ,$$

$$(45) \qquad s_1(v) = 2\alpha e^{-\frac{1}{4}\pi i - i\alpha} \int_0^\infty \frac{t^{\frac{1}{2}}(1-it)^{\frac{5}{2}}}{1+v-it} e^{-\alpha t} dt .$$

On account of (40) we have

$$p(w) \sim -\frac{e^{i\alpha w}}{\pi^2 w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \left\{ \frac{2}{\alpha^2} (\alpha^2 v^2 - i\alpha v - 2) + 2i\alpha v^3 \int_0^\infty \frac{e^{-i\alpha t}}{t-v} dt \right\} \frac{dv}{v-w}$$

$$+ \sum_{n=1} \left\{ \frac{S_n(w) e^{i\alpha w}}{w} + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_1^\infty \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \frac{S_n(t) e^{-2i\alpha t}}{t-w} dt \right\} .$$

Now

$$\frac{2}{\alpha^2}\int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \frac{\alpha^2 v^2 - i\alpha v - 2}{v - w}\,dv = \frac{2}{\alpha^2}\int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \left\{\alpha^2 v - \alpha(i - \alpha w) + \frac{\alpha^2 w^2 - i\alpha w - 2}{v - w}\right\} dv$$

$$= \frac{2\pi}{\alpha^2}\left[\frac{1}{8}\alpha^2 - \frac{1}{2}\alpha(i-\alpha w) - (\alpha^2 w^2 - i\alpha w - 2)\right]$$

and

$$\int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}} \frac{2i\alpha v^3}{v - w} \int_0^\infty \frac{e^{-i\alpha t}}{t - v}\,dt\,dv = -2i\alpha \int_0^\infty e^{-i\alpha t} \int_0^1 \left(\frac{1-v}{v}\right)^{\frac{1}{2}}$$

$$\times \left\{v + t + w + \frac{v(t^2+tw+w^2) - wt(t+w)}{(v-w)(v-t)}\right\} dv\,dt$$

$$= -2i\alpha \int_0^\infty e^{-i\alpha t}\left[\frac{1}{8}\pi + \frac{1}{2}\pi(t+w) + (t^2+tw+w^2)\right.$$

$$\times \left\{-\pi + \pi^2 w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}\delta(t-w) + \pi t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}\,\frac{H(t-1)}{t - w}\right\}$$

$$\left. - wt(t+w)\left\{\pi^2\left(\frac{1-w}{w}\right)^{\frac{1}{2}}\delta(t-w) + \pi\left(\frac{t-1}{t}\right)^{\frac{1}{2}}\,\frac{H(t-1)}{t - w}\right\}\right] dt$$

from (E6) and (E3) of Jones [17]. The last integral can be written

$$-2\pi i\alpha \int_0^1 e^{-i\alpha t}\left\{\frac{1}{8} + \frac{1}{2}(t+w) - (t^2 + tw + w^2)\right\} dt \; -2i\alpha\pi^2 w^{\frac{5}{2}}\, e^{-i\alpha w}(1-w)^{\frac{1}{2}}$$

$$-2\pi i\alpha \int_1^\infty e^{-i\alpha t}\left\{\frac{1}{8} + \frac{1}{2}(t+w) - (t^2+tw+w^2) + \frac{t^3}{t-w}\left(\frac{t-1}{t}\right)^{\frac{1}{2}}\right\} dt\,.$$

A comparison with (41) and (42) reveals that we have the same integrals with $-w$ in place of v and since no principal values are involved we can deduce that

$$p(w) \sim 2i\alpha w^2 + \frac{2\alpha e^{i\alpha(w-1) - \frac{1}{4}\pi i}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_0^\infty \frac{(1-it)^{\frac{5}{2}} t^{\frac{1}{2}} e^{-\alpha t}}{1 - w - it}\, dt$$

(46)

$$+ \sum_{n=1} \left\{ \frac{S_n(w)e^{-i\alpha w}}{w} + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_1^\infty \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \frac{S_n(t)e^{-2i\alpha t}}{t - w}\, dt \right\} .$$

With p known p_1 can be found from (38) and the whole distribution is known as soon as the constants d_o, d_1 and A_1 have been determined. We turn to this problem in the next section. Before doing that, however, we quote the following

$$f(w) \sim 2i\alpha + \frac{2\alpha e^{i\alpha(w-1) - \frac{1}{4}\pi i}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_0^\infty \frac{(1-it)^{\frac{1}{2}} t^{\frac{1}{2}}}{1 - w - it}\, e^{-\alpha t}\, dt$$

(47)

$$+ \sum_{n=1} \left\{ \frac{H_n(w)e^{-i\alpha w}}{w} + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_1^\infty \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \frac{H_n(t)e^{-2i\alpha t}}{t - w}\, dt \right\}$$

where H_n and h_n are related by equations of the type (43) - (45) and

$$h_1(v) = 2\alpha e^{-i\alpha - \frac{1}{4}\pi i} \int_0^\infty \frac{(1-it)^{\frac{1}{2}} t^{\frac{1}{2}}}{1 + v - it}\, e^{-\alpha t}\, dt .$$

5. The equations for d_o, d_1 and A_1.

It is clear from (34), (46) and (47) that d(w), p(w) and f(w) all become singular like $(1-w)^{-\frac{1}{2}}$ as $w \to 1$. Suppose that, near $w = 1$,

$$f(w) = (1-w)^{-\frac{1}{2}}\{\lambda_1 + \lambda_2(1-w) + \lambda_3(1-w)^2\} + O\{(1-w)^2\} ,$$

$$d(w) = (1-w)^{-\frac{1}{2}}\{\mu_1 + \mu_2(1-w) + \mu_3(1-w)^2\} + O\{(1-w)^2\} ,$$

$$p(w) = (1-w)^{-\frac{1}{2}}\{\nu_1 + \nu_2(1-w) + \nu_3(1-w)^2\} + O\{(1-w)^2\} .$$

It is not perhaps immediately obvious from (34), (46) and (47) that there are no terms involving the powers $(1-w)^o$ and $(1-w)$ but this is proved in Appendix C.

We have seen at the beginning of Section 4 that $p_1(w)$ must vanish like $(1-w)^{3/2}$ as $w \to 1$. Therefore (38) implies that

$$(48)\ d_o\lambda_1 - A_1\mu_1/(8\pi\alpha^9)^{\frac{1}{2}} + d_1\nu_1 = 0 ,$$

$$(49)\ d_o\lambda_2 - A_1\mu_2/(8\pi\alpha^9)^{\frac{1}{2}} + d_1\nu_2 = 0 .$$

Also, it was pointed out in Section 1 that, because of the edge conditions, $a_1(r_1) + b_1(r_1) = 0$ at $r_1 = 1$. On account of (29) and (36) this implies that

$$\frac{1}{4\pi\alpha^2}\{A_1 d(w) - f(w)\} + w^2\left(\frac{d}{wdw}\right)^2 p_1(w) = 0$$

at $w = 1$. This requires that

$$(50) \qquad \frac{1}{4\pi\alpha^2}(A_1\mu_1 - \lambda_1) + \frac{3}{4}\{d_o\lambda_3 - A_1\mu_3/(8\pi\alpha^9)^{\frac{1}{2}} + d_1\nu_3\}\left(\frac{\alpha}{2\pi}\right)^{\frac{1}{2}} = 0 .$$

Equations (48) - (50) constitute three linear equations for the constants d_o, d_1 and A_1 and hence determine them (so long as the equations are independent).

It is shown in Appendix C that

$$\lambda_1 = \frac{2\alpha}{\pi} e^{\frac{1}{4}\pi i}\int_o^\infty \frac{(1-it)^{\frac{1}{2}}}{t^{\frac{1}{2}}} e^{-\alpha t} dt + \sum_{n=1} \frac{e^{i\alpha}}{\pi}\int_1^\infty \frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}} dt,$$

$$\lambda_2 = (\tfrac{1}{2} - i\alpha)\lambda_1 + \frac{2\alpha}{\pi} e^{-\frac{1}{4}\pi i}\left\{\int_o^\infty \frac{K_1(t)}{t^{\frac{1}{2}}} e^{-\alpha t} dt - 2\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}\right\} - \sum_{n=1} \frac{2e^{i\alpha}}{\pi}\int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}}\left\{\frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}',$$

$$\lambda_3 = (\tfrac{1}{8} + \tfrac{1}{2}i\alpha + \tfrac{1}{2}\alpha^2)\lambda_1 + (\tfrac{1}{2} - i\alpha)\lambda_2 + \frac{2\alpha}{\pi} e^{\frac{1}{4}\pi i}\left\{\tfrac{2}{3}\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}} - \int_o^\infty \frac{K_1(t) + \frac{1}{2}i}{t^{\frac{3}{2}}} e^{-\alpha t} dt\right\}$$

$$+ \sum_{n=1} \frac{4e^{i\alpha}}{3\pi}\int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}}\left\{\frac{H_n(t)}{t^{\frac{1}{2}}} e^{-2i\alpha t}\right\}'',$$

$$\mu_1 = \frac{2e^{i\alpha}}{\pi} + \sum_{n=1} \frac{e^{i\alpha}}{\pi}\int_1^\infty \frac{E_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}} dt ,$$

$$\mu_2 = (\tfrac{1}{2} - i\alpha)\mu_1 - \sum_{n=1} \frac{2e^{i\alpha}}{\pi}\int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}}\left\{\frac{E_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}' ,$$

$$\mu_3 = (\tfrac{1}{8} + \tfrac{1}{2}i\alpha + \tfrac{1}{2}\alpha^2)\mu_1 + (\tfrac{1}{2} - i\alpha)\mu_2 + \sum_{n=1} \frac{4e^{i\alpha}}{3\pi}\int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}}\left\{\frac{E_n(t)}{t^{\frac{1}{2}}} e^{-2i\alpha t}\right\}'',$$

$$\nu_1 = \frac{2\alpha}{\pi} e^{\frac{1}{4}\pi i} \int_0^\infty \frac{(1-it)^{\frac{5}{2}}}{t^{\frac{1}{2}}} e^{-\alpha t} dt + \sum_{n=1} \frac{e^{i\alpha}}{\pi} \int_1^\infty \frac{S_n(t) e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}} dt,$$

$$\nu_2 = (\tfrac{1}{2} - i\alpha)\nu_1 + \frac{2\alpha e^{-\frac{1}{4}\pi i}}{\pi} \left\{ \int_0^\infty \frac{K_5(t)}{t^{\frac{1}{2}}} e^{-\alpha t} dt - 2\pi^{\frac{1}{2}} \alpha^{\frac{1}{2}} \right\} -$$

$$- \sum_{n=1} \frac{2e^{i\alpha}}{\pi} \int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}} \left\{ \frac{S_n(t) e^{-2i\alpha t}}{t^{\frac{1}{2}}} \right\}' ,$$

$$\nu_3 = (\tfrac{1}{8} + \tfrac{1}{2} i\alpha + \tfrac{1}{2}\alpha^2)\nu_1 + (\tfrac{1}{2} - i\alpha)\nu_2 + \frac{2\alpha}{\pi} e^{\frac{1}{4}\pi i} \left\{ \tfrac{2}{3}\pi^{\frac{1}{2}} \alpha^{\frac{3}{2}} - 4i\pi^{\frac{1}{2}} \alpha^{\frac{1}{2}} - \int_0^\infty \frac{K_5(t) + \frac{5}{2} i}{t^{3/2}} e^{-\alpha t} dt \right\}$$

$$+ \sum_{n=1} \frac{4e^{i\alpha}}{3\pi} \int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}} \left\{ \frac{S_n(t)}{t^{\frac{1}{2}}} e^{-2i\alpha t} \right\}''$$

where $K_m(t) = \{(1-it)^m - 1\}/t$. Since (48) is satisfied the terms involving λ_1, μ_1 and ν_1 in λ_2, μ_2 and ν_2 respectively can be ignored for (49). Because (48) and (49) are satisfied, the terms involving $\lambda_1, \lambda_2, \mu_1, \mu_2, \nu_1$ and ν_2 in λ_3, μ_3, ν_3 can be neglected as far as (50) is concerned.

The above formulas permit the calculation of $\lambda_1, \lambda_2, \ldots, \nu_3$ to any desired degree of accuracy. The first few terms of the asymptotic development are derived in Appendix D. Thus

$$(51)\quad \lambda_1 = \frac{2\alpha^{\frac{1}{2}}e^{\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}}\left(1 - \frac{i}{4\alpha} + \frac{1}{32\alpha^2} + \frac{5i}{128\alpha^3}\right) + \frac{e^{-2i\alpha}}{4\pi\alpha}\left(1 + \frac{7i}{16\alpha} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{3/2}}\right) + O(\alpha^{-3}) ,$$

$$(52)\quad \mu_1 = \frac{2e^{i\alpha}}{\pi}\left(1 - \frac{i}{8\alpha} - \frac{1}{128\alpha^2}\right) + \frac{e^{\frac{1}{4}\pi i - i\alpha}}{\pi^{\frac{3}{2}}\alpha^{\frac{1}{2}}}\left(1 + \frac{3i}{16\alpha} - \frac{113}{512\alpha^2} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{3/2}}\right) + O(\alpha^{-3}),$$

$$(53)\quad \nu_1 = \frac{2\alpha^{\frac{1}{2}}e^{\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}}\left(1 - \frac{5i}{4\alpha} - \frac{47}{32\alpha^2} + \frac{89i}{128\alpha^3}\right) + \frac{e^{-2i\alpha}}{4\pi\alpha}\left(1 - \frac{41i}{16\alpha} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{3/2}}\right) + O(\alpha^{-3}) ,$$

$$(54)\quad \lambda_2 = \left(\frac{1}{2} - i\alpha\right)\lambda_1 - \frac{4\alpha^{\frac{3}{2}}}{\pi^{\frac{1}{2}}}\, e^{-\frac{1}{4}\pi i}\left(1 + \frac{i}{4\alpha} - \frac{1}{32\alpha^2} + \frac{i}{128\alpha^3}\right) + \frac{ie^{-2i\alpha}}{\pi}\left(1 + \frac{i}{16\alpha} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{3/2}}\right) + O(\alpha^{-2}),$$

$$(55)\quad \mu_2 = \left(\frac{1}{2} - i\alpha\right)\mu_1 - \frac{ie^{i\alpha}}{4\pi\alpha} + \frac{4i\alpha^{\frac{1}{2}}}{\pi^{3/2}}\, e^{\frac{1}{4}\pi i - i\alpha}\left(1 + \frac{i}{16\alpha} - \frac{45}{512\alpha^2} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{3/2}}\right) + O(\alpha^{-2}),$$

$$(56)\quad \nu_2 = \left(\frac{1}{2} - i\alpha\right)\nu_1 - \frac{4\alpha^{\frac{3}{2}}}{\pi^{\frac{1}{2}}}\, e^{-\frac{1}{4}\pi i}\left(1 + \frac{5i}{4\alpha} + \frac{15}{32\alpha^2} - \frac{11i}{128\alpha^3}\right) + \frac{ie^{-2i\alpha}}{\pi}\left(1 - \frac{47i}{16\alpha} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{3/2}}\right) + O(\alpha^{-2}),$$

$$(57)\quad \lambda_3 = \left(\frac{1}{8} + \frac{1}{2}i\alpha + \frac{1}{2}\alpha^2\right)\lambda_1 + \left(\frac{1}{2} - i\alpha\right)\lambda_2 - \frac{8\alpha^{\frac{5}{2}}e^{\frac{1}{4}\pi i}}{3\pi^{\frac{1}{2}}}\left(1 + \frac{3i}{4\alpha} + \frac{3}{32\alpha^2} + \frac{3i}{128\alpha^3}\right)$$

$$- \frac{4\alpha e^{-2i\alpha}}{3\pi}\left(1 - \frac{5i}{16\alpha} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{3/2}}\right) + O(\alpha^{-1}),$$

$$(58)\quad \mu_3 = \left(\frac{1}{8} + \frac{1}{2}i\alpha + \frac{1}{2}\alpha^2\right)\mu_1 + \left(\frac{1}{2} - i\alpha\right)\mu_2 - \frac{16\alpha^{3/2}}{3\pi^{3/2}}\, e^{\frac{1}{4}\pi i - i\alpha}\left(1 - \frac{i}{16\alpha} - \frac{17}{512\alpha^2} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{3/2}}\right) + O(\alpha^{-1})$$

$$(59)\quad \nu_3 = \left(\frac{1}{8} + \frac{1}{2}i\alpha + \frac{1}{2}\alpha^2\right)\nu_1 + \left(\frac{1}{2} - i\alpha\right)\nu_2 - \frac{8\alpha^{\frac{5}{2}}}{3\pi^{\frac{1}{2}}}\, e^{\frac{1}{4}\pi i}\left(1 + \frac{15i}{4\alpha} - \frac{45}{32\alpha^2} + \frac{15i}{128\alpha^3}\right)$$

$$- \frac{4\alpha}{3\pi}\, e^{-2i\alpha}\left(1 - \frac{53i}{16\alpha} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{3/2}}\right) + O(\alpha^{-1}) .$$

Substitute (51) - (56) in (48) and (49). A simpler equation is obtained by forming the combination (48) + $(i/4\alpha)$(49). Then, if (49) and (48) + $(i/4\alpha)$(49) be solved for d_o and d_1 we find

$$(60)\quad d_o\left(1-\frac{i}{\alpha}-\frac{17}{64\alpha^2}\right) = -\frac{A_1}{4(8\pi\alpha^9)^{\frac{1}{2}}}\left\{\frac{2\alpha^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\, e^{\frac{1}{4}\pi i+i\alpha}\left(1+\frac{9i}{8\alpha}+\frac{87}{128\alpha^2}\right)\right.$$

$$\left. -\frac{ie^{-i\alpha}}{\pi}\left(1-\frac{53i}{16\alpha}-\frac{737}{512\alpha^2}\right)-\frac{e^{\frac{1}{4}\pi i-3i\alpha}}{16\pi^{\frac{1}{2}}\alpha^{3/2}}\right\},$$

$$(61)\quad d_1\left(1-\frac{i}{\alpha}-\frac{17}{64\alpha^2}\right) = \frac{A_1}{4(8\pi\alpha^9)^{\frac{1}{2}}}\left\{\frac{2\alpha^{\frac{1}{2}}}{\pi^{\frac{1}{2}}}\, e^{\frac{1}{4}\pi i+i\alpha}\left(1+\frac{i}{8\alpha}+\frac{7}{128\alpha^2}\right)\right.$$

$$\left. -\frac{ie^{-i\alpha}}{\pi}\left(1-\frac{5i}{16\alpha}+\frac{127}{512\alpha^2}\right)-\frac{e^{\frac{1}{4}\pi i-3i\alpha}}{16\pi^{\frac{1}{2}}\alpha^{3/2}}\right\}.$$

Next subtract $4i\alpha/3$ times (49) from the term in { } in (50) and use (54)-(59). Then substitute for d_o and d_1 from (60) and (61). The result is

$$\frac{4}{\pi}A_1 e^{i\alpha}\left\{1+\frac{7i}{8\alpha}+\frac{e^{\frac{1}{4}\pi i-2i\alpha}}{2\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}}\left(1-\frac{13i}{16\alpha}\right)\right\} = \frac{2\alpha^{\frac{1}{2}}e^{\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}}\left(1-\frac{i}{4\alpha}\right)+\frac{e^{-2i\alpha}}{4\pi\alpha}.$$

Hence

$$(62)\quad A_1 = \frac{1}{2}\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}e^{\frac{1}{4}\pi i-i\alpha}\left\{1-\frac{9i}{8\alpha}-\frac{e^{\frac{1}{4}\pi i-2i\alpha}}{2\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}}\left(1-\frac{41i}{16\alpha}\right)+\frac{e^{\frac{1}{2}\pi i-4i\alpha}}{4\pi\alpha}-\frac{e^{\frac{3}{4}\pi i-6i\alpha}}{8\pi^{\frac{3}{2}}\alpha^{\frac{3}{2}}}\right\}+O(\alpha^{-\frac{3}{2}}).$$

Equations (60) and (61) can now be used to determine d_o and d_1 but these constants will not be required for the subsequent analysis, although they are necessary to specify the distribution on the disc completely via (38).

6. The scattering coefficient.

It is known (see, for example, Jones [23]) that, if an arbitrary obstacle irradiated by the plane wave with $\underline{E}_o = \underline{e}_o e^{-ikR\underline{n}_o \cdot \underline{n}}$ produces a scattered electric intensity $\underline{E}_s \sim \underline{A}_s(\underline{n})e^{-ikR}/R$ in the direction of the unit vector $\underline{n}$, the scattering coefficient is $-\frac{4\pi}{kS_o}\Re\,\underline{A}_s(\underline{n}_o)\cdot\underline{e}_o$ where S_o is the area of the incident wavefront intercepted by the obstacle. Translated to our coordinates in which the radius of the disc is unity this states that, if the plane wave $\underline{E}_o = \underline{e}_o e^{-i\alpha R\underline{n}_o \cdot \underline{n}}$ produces the scattered field $\underline{E}_s \sim \underline{A}_s(\underline{n})e^{-i\alpha R}/R$ the scattering coefficient of the circular disc is $-\frac{4}{\alpha}\Re\,\underline{A}_s(\underline{n}_o)\cdot\underline{e}_o$.

Now it follows from the representation (1) that

$$\underline{E}_s \sim -\alpha^2 \frac{e^{-i\alpha R}}{R}\,\underline{n} \wedge \left\{\underline{n} \wedge \int_S \underline{J}(\underline{R}_1)e^{i\alpha\underline{n}\cdot\underline{R}_1}\,dS\right\}$$

i.e.

$$\underline{A}_s(\underline{n}) = -\alpha^2\underline{n} \wedge \left\{\underline{n} \wedge \int_S \underline{J}(\underline{R})e^{i\alpha\underline{n}\cdot\underline{R}_1}\,dS\right\}.$$

But

$$\underline{n}_o \equiv (\sin\Theta_o, 0, \cos\Theta_o),\quad \underline{e}_o \equiv (\cos\varphi_o\cos\Theta_o,\ \sin\varphi_o,\ -\cos\varphi_o\sin\Theta_o)$$

and so

$$\underline{A}_s(\underline{n}_o)\cdot\underline{e}_o = \alpha^2\int_0^1\int_0^{2\pi}\{(J_r\cos\varphi - J_\varphi\sin\varphi)\cos\varphi_o\cos\Theta_o + (J_r\sin\varphi + J_\varphi\cos\varphi)\sin\varphi_o\} \times e^{-i\alpha r_1\sin\Theta_o\cos\varphi}\,r_1\,dr_1\,d\varphi\,.$$

Substitute for J_r and J_φ from (3) and use the formulae

$$\int_0^{2\pi} e^{im\varphi+i\alpha r_1 \sin\theta_0 \cos\varphi}\, d\varphi = 2\pi e^{\frac{1}{2}m\pi i} J_m(\alpha r_1 \sin\theta_0),$$

$$J_{\nu-1}(z) + J_{\nu+1}(z) = 2\nu J_\nu(z)/z, \quad J_{\nu-1}(z) - J_{\nu+1}(z) = 2J'_\nu(z).$$

Then

$$\underline{A}_s(\underline{n}_0)\cdot\underline{e}_0 = \pi\alpha^2 \sum_{n=-\infty}^{\infty} e^{\frac{1}{2}(n+1)\pi i} \int_0^1 \left[\left\{ (a_n - b_n) \frac{nJ_n(\alpha r_1 \sin\theta_0)}{\alpha r_1 \sin\theta_0} - (a_n + b_n) J'_n(\alpha r_1 \sin\theta_0) \right\} \cos\theta_0 \cos\varphi_0 \right.$$
$$\left. - i \left\{ (a_n + b_n) \frac{nJ_n(\alpha r_1 \sin\theta_0)}{\alpha r_1 \sin\theta_0} - (a_n - b_n) J'_n(\alpha r_1 \sin\theta_0) \right\} \sin\varphi_0 \right] r_1 dr_1 \tag{63}$$

from which the scattering coefficient can be determined.

In the particular case of the normally incident plane wave $\theta_0 = \varphi_0 = 0$ and (63) reduces to

$$\underline{A}_s(\underline{n}_0)\cdot\underline{e}_0 = \pi\alpha^2 \int_0^1 (b_1 + a_{-1}) r_1 dr_1 .$$

Since $a_{-1} = b_1$ the scattering coefficient σ is given by

$$\sigma = -8\pi\alpha \Im \int_0^1 b_1(w) w dw$$

or, from (29),

$$\sigma = \frac{2}{\alpha} \Im \int_0^1 \{f(w) - A_1 d(w)\}\, w dw . \tag{64}$$

7. The scattering coefficient at high frequencies.

The quantity $(2/\alpha)\vartheta \int_0^1 wf(w)dw$ is the scattering coefficient of the sound-soft disc in a normally incident plane wave. Its value at high frequencies has already been calculated (Jones [17]) and it may be denoted by σ_{soft}. Thus

$$\sigma = \sigma_{soft} - \frac{2}{\alpha}\vartheta A_1 \int_0^1 wd(w)dw \ . \tag{65}$$

For $\alpha \gg 1$, $d(w)$ is given by (34). The first term in the expansion of $d(w)$ gives a contribution to the integral in σ of σ_0' where

$$\begin{aligned} \sigma_0' &= \int_0^1 \frac{2w^{\frac{1}{2}}e^{i\alpha w}}{\pi(1-w)^{\frac{1}{2}}}\, dw \\ &= \frac{2e^{i\alpha - \frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}}\left(1 + \frac{i}{4\alpha}\right) + \frac{e^{\frac{3}{4}\pi i}}{\pi^{\frac{1}{2}}\alpha^{3/2}} + O(\alpha^{-\frac{5}{2}}) \ . \end{aligned} \tag{66}$$

The general term of the series for $d(w)$ gives a contribution to σ of $-\frac{2}{\alpha}\vartheta A_1\sigma_n'$ where

$$\sigma_n' = \int_0^1 \left\{ E_n(w)e^{-i\alpha w} + \frac{w^{\frac{1}{2}}e^{i\alpha w}}{\pi(1-w)^{\frac{1}{2}}} \int_1^\infty \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \frac{E_n(t)e^{-2i\alpha t}}{t - w}\, dt \right\} dw \ .$$

The asymptotic behavior of σ_n' may be determined by the same procedure as was used for the corresponding quantity in the scalar problem. All that is necessary to put e_n for h_n. Consequently

$$
\begin{aligned}
\sigma'_n = {} & \frac{e^{-i\alpha}}{2\pi\alpha}\left\{-e_n(1)+\frac{7i}{16\alpha}\,e_n(1)+\frac{5i}{4\alpha}\,e'_n(1)-\frac{2^{\frac{1}{2}}i}{8\alpha}\,e_n(0)\right\} \\
& + \frac{e^{-\frac{1}{4}\pi i}}{2\pi^{\frac{1}{2}}\alpha^{3/2}}\left[e_n(0)+\frac{i}{\alpha}\left\{\frac{13}{16}\,e_n(0)-\frac{3}{2}\,e'_n(0)\right\}\right] \qquad (67) \\
& - \frac{e^{\frac{1}{4}\pi i+i\alpha}}{8\pi^{\frac{1}{2}}\alpha^{3/2}}\left[e_n(0)-\frac{i}{4\alpha}\left\{e_n(0)+\frac{9}{4}\,e'_n(0)\right\}\right] + 0\left(\frac{e_n}{\alpha^4}\right) .
\end{aligned}
$$

Because of (D16),

$$
(68)\quad \sigma'_1 = \frac{e^{-i\alpha}}{\pi\alpha}\left(-1+\frac{7i}{16\alpha}-\frac{2^{\frac{1}{2}}i}{8\alpha}\right)+\frac{e^{-\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}\alpha^{3/2}}-\frac{e^{\frac{1}{4}\pi i+i\alpha}}{4\pi^{\frac{1}{2}}\alpha^{3/2}}+0(\alpha^{-\frac{5}{2}})
$$

and, from (D11),

$$
(69)\quad \sigma'_2 = -\frac{ie^{-i\alpha}}{8\pi\alpha^2}\left(1-2^{\frac{1}{2}}\right)+0(\alpha^{-\frac{5}{2}}) ,
$$

$$
(70)\quad \sigma'_n = 0(\alpha^{-n}) \qquad\qquad (n \geq 1) .
$$

Combining (66), (68) - (70) we have

$$
\int_0^1 wd(w)dw = \frac{2e^{i\alpha-\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}}\left(1+\frac{i}{8\alpha}\right)-\frac{e^{-i\alpha}}{\pi\alpha}\left(1-\frac{5i}{16\alpha}\right)+0(\alpha^{-\frac{5}{2}}) .
$$

Hence, from (63) and (65),

$$\sigma = \sigma_{soft} - \frac{2}{\alpha} \Im \left\{ 1 - \frac{e^{\frac{1}{4}\pi i - 2i\alpha}}{2\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}} \left(1 - \frac{41i}{16\alpha}\right) - \frac{9i}{8\alpha} + \frac{e^{\frac{1}{2}\pi i - 4i\alpha}}{4\pi\alpha} - \frac{e^{\frac{3}{4}\pi i - 6i\alpha}}{8\pi^{3/2}\alpha^{3/2}} \right\}$$

$$\times \left\{ 1 + \frac{i}{8\alpha} - \frac{e^{\frac{1}{4}\pi i - 2i\alpha}}{2\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}} \left(1 - \frac{5i}{16\alpha}\right) \right\} + O(\alpha^{-3})$$

$$= \sigma_{soft} - \frac{2}{\alpha} \Im \left\{ 1 - \frac{i}{\alpha} - \frac{e^{\frac{1}{4}\pi i - 2i\alpha}}{\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}} \left(1 - \frac{31i}{16\alpha}\right) + \frac{e^{\frac{1}{2}\pi i - 4i\alpha}}{2\pi\alpha} - \frac{e^{\frac{3}{4}\pi i - 6i\alpha}}{4\pi^{3/2}\alpha^{3/2}} \right\} + O(\alpha^{-3})$$

Since

$$\sigma_{soft} = 2 \left\{ 1 - \frac{1}{4\alpha^2} + \frac{\cos(2\alpha - \frac{1}{4}\pi)}{4\pi^{\frac{1}{2}}\alpha^{5/2}} \right\} + O(\alpha^{-3})$$

it follows that

$$\sigma = 2 \left\{ 1 - \frac{1}{\pi^{\frac{1}{2}}\alpha^{3/2}} \sin(2\alpha - \tfrac{1}{4}\pi) + \frac{3}{4\alpha^2} + \frac{1}{2\pi\alpha^2} \sin(4\alpha - \tfrac{1}{4}\pi) \right. \tag{71}$$

$$\left. - \frac{1}{4\pi^{3/2}\alpha^{5/2}} \sin(6\alpha - \tfrac{3}{4}\pi) - \frac{27}{16\pi^{\frac{1}{2}}\alpha^{5/2}} \cos(2\alpha - \tfrac{1}{4}\pi) \right\} + O(\alpha^{-3}).$$

This scattering coefficient agrees with that found by Seshradi and Wu [15] except for the last term. The numerical factor of the last term is found by them to be 7/4 instead of 27/16; the difference is probably due to an arithmetical slip on one of our parts but this has not, so far, been located.

Since all the high-frequency integral equations solved here are of the type for which uniqueness and an asymptotic development were proved, the same properties are valid here. Consequently, it would be possible to make an estimate of the magnitude of the order term in (71) though it would be rather complicated because of the intervention of d_0, d_1 and A_1. Experience indicates that a formula such as (71) may be used for numerical values when $\alpha \geq 3$ and will give tolerably accurate results even for an α as low as 2.

APPENDIX A

In this Appendix is derived the formula for

$$\psi_n(r,r_1) = \int_0^{2\pi} \frac{e^{-i\alpha(r^2+r_1^2-2rr_1\cos\varphi)^{\frac{1}{2}}}}{(r^2+r_1^2-2rr_1\cos\varphi)^{\frac{1}{2}}} \cos n\varphi d\varphi .$$

We deal firstly with the imaginary past. Now

$$\int_0^{2\pi} \frac{\sin\{\alpha(r^2+r_1^2-2rr_1\cos\varphi)^{\frac{1}{2}}\}}{(r^2+r_1^2-2rr_1\cos\varphi)^{\frac{1}{2}}} \cos n\varphi d\varphi = (\tfrac{1}{2}\alpha\pi)^{\frac{1}{2}} \int_0^{2\pi} \frac{J_{\frac{1}{2}}\{\alpha(r^2+r_1^2-2rr_1\cos\varphi)^{\frac{1}{2}}\}}{(r^2+r_1^2-2rr_1\cos\varphi)^{\frac{1}{2}}} \cos n\varphi d\varphi$$

$$= \alpha \int_0^{2\pi}\int_0^{\frac{1}{2}\pi} \sin\Theta\, J_0\{\alpha(r^2+r_1^2-2rr_1\cos\varphi)^{\frac{1}{2}}\sin\Theta\} \cos n\varphi\, d\Theta\, d\varphi$$

after using Sonine's first finite integral for Bessel functions (Watson [24]). The addition theorem for Bessel functions gives

$$-\Im\psi_n = \alpha \int_0^{2\pi}\int_0^{\frac{1}{2}\pi} \sin\Theta \sum_{m=0}^{\infty} \epsilon_m J_m(\alpha r\sin\Theta)\, J_m(\alpha r_1\sin\Theta) \cos m\varphi \cos n\varphi d\Theta d\varphi$$

where $\epsilon_0 = 1$, $\epsilon_m = 2$ $(m \neq 0)$. Hence

$$-\partial \psi_n = 2\pi\alpha \int_0^{\frac{1}{2}\pi} J_n(\alpha r \sin \theta)\, J_n(\alpha r_1 \sin \theta) \sin \theta \, d\theta \, .$$

Since

$$\left(\frac{d}{zdz}\right)^m \frac{J_\nu(z)}{z^\nu} = (-1)^m \frac{J_{\nu+m}(z)}{z^{\nu+m}} \, , \qquad \left(\frac{d}{zdz}\right)^m \frac{J_\nu(az)}{z^\nu} = (-)^m a^m \frac{J_{\nu+m}(az)}{z^{\nu+m}}$$

and therefore

$$-\partial \psi_n = 2\pi(-)^n r_1^n \left(\frac{d}{r_1 dr_1}\right)^n \alpha^{1-n} \int_0^{\frac{1}{2}n} J_n(\alpha r \sin \theta)\, J_o(\alpha r_1 \sin\theta) \sin^{1-n}\theta d\theta.$$

It has been shown elsewhere (Jones [22]) that

$$\int_0^{\frac{1}{2}\pi} J_{\nu-\frac{1}{2}}(z \sin \theta) J_{\mu+\frac{1}{2}}(Z \sin \theta) \sin^{\nu-\mu}\theta \cos^{2\lambda}\theta \, d\theta$$

$$= z^{\nu-\frac{1}{2}} Z^{\lambda+\frac{1}{2}} \int_0^{\frac{1}{2}\pi} J_{\mu-\lambda}(Z \sin \theta) \frac{J_{\nu+\lambda}\{(z^2 - Z^2\cos^2\theta)^{\frac{1}{2}}\}}{(z^2 - Z^2\cos^2\theta)^{\frac{1}{2}(\nu+\lambda)}} \sin^{\mu-\lambda+1}\theta \cos^{2\lambda}\theta d\theta$$

when $\mathcal{R}(\lambda) > -\frac{1}{2}$, $\mathcal{R}(\mu) > \mathcal{R}(\lambda) - 1$, $\mathcal{R}(\nu) > -\frac{1}{2}$. Use this formula with $\lambda=0$, $\mu = n - \frac{1}{2}$, $\nu = \frac{1}{2}$, $Z = \alpha r$, $z = \alpha r_1$ and then

$$-\partial\psi_n = 2\pi(-)^n r_1^n \left(\frac{d}{r_1 dr_1}\right)^n \alpha^{1-n} (\alpha r)^{\frac{1}{2}} \int_0^{\frac{1}{2}\pi} J_{n-\frac{1}{2}}(\alpha r \sin \theta) \frac{J_{\frac{1}{2}}\{\alpha(r_1^2 - r^2\cos^2\theta)^{\frac{1}{2}}\}}{\alpha^{\frac{1}{2}}(r_1^2 - r^2\cos^2\theta)^{1/4}} \sin^{n+\frac{1}{2}} \theta d\theta$$

(A1)

$$= 2\pi(-)^n \frac{r_1^n}{\alpha^{n-1} r^n} \left(\frac{d}{r_1 dr_1}\right)^n \int_0^r J_{n-\frac{1}{2}}\{\alpha(r^2-x^2)^{\frac{1}{2}}\} \frac{J_{\frac{1}{2}}\{\alpha(r_1^2-x^2)^{\frac{1}{2}}\}}{(r_1^2-x^2)^{1/4}} (r^2-x^2)^{\frac{1}{2}(n-\frac{1}{2})} \, dx$$

on putting $r \cos \theta = x$.

Turning now to the real part we employ the result (Watson [24]),

$$\frac{\cos by}{b} = \int_y^\infty \frac{J_o(bt)t}{(t^2-y^2)^{\frac{1}{2}}}\, dt$$

and then

$$\mathfrak{R}\psi_n = \int_o^{2\pi}\int_\alpha^\infty J_o\{r^2+r_1^2-2rr_1\cos\varphi)^{\frac{1}{2}}t\}\,\frac{t}{(t^2-\alpha^2)^{\frac{1}{2}}}\cos n\varphi\, dt\, d\varphi$$

$$= \int_\alpha^\infty \frac{t}{(t^2-\alpha^2)^{\frac{1}{2}}}\int_o^{2\pi}\sum_{m=0}^\infty \epsilon_m J_m(tr)J_m(tr_1)\cos m\varphi\cos n\varphi\, d\varphi\, dt$$

$$= 2\pi\int_\alpha^\infty \frac{t}{(t^2-\alpha^2)^{\frac{1}{2}}}\, J_n(tr)J_n(tr_1)dt$$

$$= 2\pi(-)^n\, r_1^n\Big(\frac{d}{r_1dr_1}\Big)^n\int_\alpha^\infty \frac{t^{1-n}}{(t^2-\alpha^2)^{\frac{1}{2}}}\, J_n(tr)J_o(tr_1)dt$$

$$= 2\pi(-)^n r_1^n\Big(\frac{d}{r_1dr_1}\Big)^n\int_o^\infty \frac{J_n\{(t^2+\alpha^2)^{\frac{1}{2}}r\}}{(t^2+\alpha^2)^{\frac{1}{2}n}}\, J_o\{(t^2+\alpha^2)^{\frac{1}{2}}r_1\}\, dt\ .$$

According to Sonine's second finite integral (Watson [24])

$$\frac{J_n\{(t^2+\alpha^2)^{\frac{1}{2}}r\}}{(t^2+\alpha^2)^{\frac{1}{2}n}} = r^n\int_o^{\frac{1}{2}\pi}\frac{J_{-\frac{1}{2}}(tr\sin\Theta)}{(tr)^{-\frac{1}{2}}}\ \frac{J_{n-\frac{1}{2}}(\alpha r\cos\Theta)}{(\alpha r)^{n-\frac{1}{2}}}\ \sin^{\frac{1}{2}}\Theta\,\cos^{n+\frac{1}{2}}\Theta d\Theta$$

and hence

$$\mathcal{R}\psi_n = 2\pi(-)^n r_1^n \left(\frac{d}{r_1 dr_1}\right)^n r\alpha^{\frac{1}{2}-n} \int_0^{\frac{1}{2}\pi} J_{n-\frac{1}{2}}(\alpha r \cos\Theta)\sin^{\frac{1}{2}}\Theta \cos^{n+\frac{1}{2}}\Theta \times$$

$$\int_0^\infty t^{\frac{1}{2}} J_{-\frac{1}{2}}(tr\sin\Theta) J_o\{(t^2+\alpha^2)^{\frac{1}{2}} r_1\} dt\, d\Theta.$$

Now

$$\int_0^\infty t^{\frac{1}{2}} J_{-\frac{1}{2}}(bt) J_o\{a(t^2+\alpha^2)^{\frac{1}{2}}\} dt = \begin{cases} 0 & (a < b) \\ \dfrac{1}{b^{\frac{1}{2}}} \left\{\dfrac{\alpha}{(a^2-b^2)^{\frac{1}{2}}}\right\} J_{-\frac{1}{2}}\{\alpha(a^2-b^2)^{\frac{1}{2}}\} & (a > b) \end{cases}$$

and therefore

$$\mathcal{R}\psi_n = 2\pi(-)^n r_1^n \left(\frac{d}{r_1 dr_1}\right)^n r^{\frac{1}{2}}\alpha^{1-n} \int_0 J_{n-\frac{1}{2}}(\alpha r\cos\Theta)\cos^{n+\frac{1}{2}}\Theta \frac{J_{-\frac{1}{2}}\{\alpha(r_1^2-r^2\sin^2\Theta)^{\frac{1}{2}}\}}{(r_1^2-r^2\sin^2\Theta)^{1/4}} d\Theta$$

the upper limit of integration being $\frac{1}{2}\pi$ if $r_1 \geq r$ and $\sin^{-1}(r_1/r)$ if $r_1 \leq r$.

The substitution $r \sin\Theta = x$ gives

(A2)
$$\mathcal{R}\psi_n = 2\pi(-)^n \frac{r_1^n}{\alpha^{n-1} r^n} \left(\frac{d}{r_1 dr_1}\right)^n \int_0^{\min(r,r_1)} J_{n-\frac{1}{2}}\{\alpha(r^2-x^2)^{\frac{1}{2}}\} \frac{J_{-\frac{1}{2}}\{\alpha(r_1^2-x^2)^{\frac{1}{2}}\}}{(r_1^2-x^2)^{1/4}} (r^2-x^2)^{\frac{1}{2}(n-\frac{1}{2})} dx .$$

Combining (A1) and (A2) we obtain

$$\psi_n = 2\pi(-)^n \frac{r_1^n}{\alpha^{n-1} r^n} \left(\frac{d}{r_1 dr_1}\right)^n \Bigg[\int_0^{\min(r,r_1)} J_{n-\frac{1}{2}}\{\alpha(r^2-x^2)^{\frac{1}{2}}\} J_{-\frac{1}{2}}\{\alpha(r_1^2-x^2)^{\frac{1}{2}}\} \frac{(r^2-x^2)^{\frac{1}{2}(n-\frac{1}{2})}}{(r_1^2-x^2)^{1/4}} dx$$

(A3)
$$- i\int_0^r J_{n-\frac{1}{2}}\{\alpha(r^2-x^2)^{\frac{1}{2}}\} J_{\frac{1}{2}}\{\alpha(r_1^2-x^2)^{\frac{1}{2}}\} \frac{(r^2-x^2)^{\frac{1}{2}(n-\frac{1}{2})}}{(r_1^2-x^2)^{1/4}} dx\Bigg].$$

Since ψ_n is symmetric in r and r_1 we can interchange r and r_1 on the right-hand side of (A3) and it will still hold, i.e.,

$$\psi_n = 2\pi(-)^n \frac{r^n}{\alpha^{n-1} r_1^n} \left(\frac{d}{rdr}\right)^n \left[\int_0^{\min(r,r_1)} J_{n-\frac{1}{2}}\{\alpha(r_1^2-x^2)^{\frac{1}{2}}\} J_{-\frac{1}{2}}\{\alpha(r^2-x^2)^{\frac{1}{2}}\} \frac{(r_1^2-x^2)^{\frac{1}{2}(n-\frac{1}{2})}}{(r^2-x^2)^{1/4}} dx \right.$$

(A4)

$$\left. - i \int_0^{r_1} J_{n-\frac{1}{2}}\{\alpha(r_1^2-x^2)^{\frac{1}{2}}\} J_{\frac{1}{2}}\{\alpha(r^2-x^2)^{\frac{1}{2}}\} \frac{(r_1^2-x^2)^{\frac{1}{2}(n-\frac{1}{2})}}{(r^2-x^2)^{1/4}} dx \right].$$

APPENDIX B

In this Appendix we indicate another approach to (15) from that given in the main text. Since, for $n \geq 1$,

$$\left(\frac{d}{rdr}\right)^n \int_0^r \frac{(u^2-r^2)^{n-1}}{(n-1)!2^{n-1}} \frac{g(u)}{u^{n-1}} du = \left(\frac{d}{rdr}\right)^{n-1} \int_0^r \frac{(-)(u^2-r^2)^{n-2}}{(n-2)!2^{n-2}} \frac{g(u)}{u^{n-1}} du$$

$$= \frac{d}{rdr} \int_0^r (-)^{n-1} \frac{g(u)}{u^{n-1}} du$$

$$= (-)^{n-1} g(r)/r^n$$

(15) is equivalent to

$$\int_0^r \frac{J_{-\frac{1}{2}}\{\alpha(r^2-x^2)^{\frac{1}{2}}\}}{(r^2-x^2)^{1/4}}\, h(x)dx - i\int_0^1 \frac{J_{\frac{1}{2}}\{\alpha(r^2-x^2)^{\frac{1}{2}}\}}{(r^2-x^2)^{1/4}}\, h(x)dx$$

(B1)

$$= \sum_{m=0}^{n-1} C_m r^{2m} - \frac{\alpha^{n-1}}{(n-1)!\,2^n\pi}\int_0^r \frac{(u^2-r^2)^{n-1}}{u^{n-1}}\, g(u)\, du$$

where $C_0, C_1, \ldots, C_{n-1}$ are arbitrary constants. If $n = 0$, the right hand side is replaced by $\alpha g(r)/2\pi$.

Replace the Bessel functions by their trigonometric equivalents and then (B1) becomes

$$\int_0^r \frac{\cos\{\alpha(r^2-x^2)^{\frac{1}{2}}\}}{(r^2-x^2)^{\frac{1}{2}}}\, h(x)dx - i\int_0^1 \frac{\sin\{\alpha(r^2-x^2)^{\frac{1}{2}}\}}{(r^2-x^2)^{\frac{1}{2}}}\, h(x)dx$$

$$= \sum_{m=0}^{n=1} C_m' r^{2m} - \frac{\alpha^{n-\frac{1}{2}}}{(n-1)!\,2^{n+\frac{1}{2}}\pi^{\frac{1}{2}}}\int_0^r \frac{(u^2-r^2)^{n-1}}{u^{n-1}}\, g(u)du$$

where $C_m' = (\pi\alpha/2)^{\frac{1}{2}} C_m$. Make the change of variables $r^2 = 1 - z$, $x^2 = 1 - t$, with the result

$$\int_z^1 \frac{\cos\{\alpha(t-z)^{\frac{1}{2}}\}}{(t-z)^{\frac{1}{2}}}\, h_1(t)dt - i\int_0^1 \frac{\sin\{\alpha(t-z)^{\frac{1}{2}}\}}{(t-z)^{\frac{1}{2}}}\, h_1(t)dt$$

(B2)

$$= \sum_{m=0}^{n-1} C_m'(1-z)^m - \frac{\alpha^{n-\frac{1}{2}}}{(n-1)!\,2^{n+\frac{1}{2}}\pi^{\frac{1}{2}}}\int_z^1 \frac{(z-u)^{n-1}}{(1-u)^{\frac{1}{2}n}}\, g\{(1-u)^{\frac{1}{2}}\}du$$

where $h_1(t) = h\{(1-t)^{\frac{1}{2}}\}/2(1-t)^{\frac{1}{2}}$.

Equation (B2) is of the same type as (24) and so

$$\text{(B3)} \qquad \int_0^1 wh_1(w^2)\left\{\frac{e^{-i\alpha(w+v)}}{w+v} + \frac{e^{-i\alpha(w-v)}}{w-v}\right\} dw = H^{(o)}(v)$$

where

$$H^{(o)}(v) = \frac{1}{2}\frac{d}{dv}\int_0^{v^2}\frac{\cos\{\alpha(v^2-z)^{\frac{1}{2}}\}}{(v^2-z)^{\frac{1}{2}}}\left[\sum_{m=0}^{n-1} C_m'(1-z)^m - \frac{\alpha^{n-\frac{1}{2}}}{(n-1)!\,2^{n+\frac{3}{2}}\pi^{\frac{1}{2}}}\int_z^1 \frac{(z-u)^{n-1}}{(1-u)^{\frac{1}{2}n}} g\{(1-u)^{\frac{1}{2}}\}du\right]dz.$$

The behavior of h_1, and hence of h, for large α, may now be determined. The constants C_m' have to be chosen so that $h(x) \to 0$ like $(1-x)^n$ as $x \to 1$ because of (16), when it is remembered that $k(r_1)$ is nearly a constant multiple of $(1-r_1)^{-\frac{1}{2}}$ as $r_1 \to 1$. Thus the C_m' are chosen so that $h_1(t) \to 0$ like t^n as $t \to 0$.

Over h is known k is found from (16) as follows. Since

$$\frac{d}{zdz}\{z^{\nu}J_{\nu}(z)\} = z^{\nu-1}J_{\nu-1}(z),$$

$$\frac{d}{xdx}\left[(r_1^2-x^2)^{\frac{1}{2}(n-\frac{1}{2})}J_{n-\frac{1}{2}}\{\alpha(r_1^2-x^2)^{\frac{1}{2}}\}\right] = -\alpha(r_1^2-x^2)^{\frac{1}{2}(n-\frac{3}{2})}J_{n-\frac{3}{2}}\{\alpha(r_1^2-x^2)^{\frac{1}{2}}\}.$$

Hence

$$\left(\frac{d}{xdx}\right)^n h(x) = (-)^n \alpha^n \int_x^1 J_{-\frac{1}{2}}\{\alpha(r_1^2-x^2)^{\frac{1}{2}}\}\ (r_1^2-x^2)^{-1/4} k(r_1) r_1^{1-n} dr_1$$

$$= (-)^n \alpha^{n-\frac{1}{2}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_x^1 \frac{\cos\{\alpha(r_1^2-x^2)^{\frac{1}{2}}\}}{(r_1^2-x^2)^{\frac{1}{2}}}\ \frac{k(r_1)}{r_1^{n-1}}\ dr_1$$

from which can be deduced that

$$\text{(B4)} \qquad \frac{k(r_1)}{r_1^{n-1}} = \frac{(-)^{n+1}}{\alpha^{n-\frac{1}{2}}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \frac{d}{dr_1} \int_{r_1}^1 \frac{x \cosh\{\alpha(x^2-r_1^2)^{\frac{1}{2}}\}}{(x^2-r_1^2)^{\frac{1}{2}}} \left(\frac{d}{xdx}\right)^n h(x)dx\ .$$

An alternative way of writing this can be obtained from the fact that

$$\frac{d}{r_1 dr_1} \int_{r_1}^1 \frac{x \cosh\{\alpha(x^2-r_1^2)^{\frac{1}{2}}\}}{(x^2-r_1^2)^{\frac{1}{2}}} \left(\frac{d}{xdx}\right)^{n-1} h(x)dx$$

$$= \frac{1}{r_1} \frac{d}{dr_1} \left[\frac{1}{\alpha} \sin h\ \{\alpha(x^2-r_1^2)^{\frac{1}{2}}\} \left(\frac{d}{xdx}\right)^{n-1} h(x) \right]_{r_1}^1 - \frac{1}{r_1} \frac{d}{dr_1} \int_{r_1}^1 \frac{x}{\alpha} \sinh\{\alpha(x^2-r_1^2)^{\frac{1}{2}}\}$$

$$\times \left(\frac{d}{xdx}\right)^n h(x)dx$$

$$= \int_{r_1}^1 \frac{x \cosh\{\alpha(x^2-r_1^2)^{\frac{1}{2}}\}}{(x^2-r_1^2)^{\frac{1}{2}}} \left(\frac{d}{xdx}\right)^n h(x)dx$$

since $h(x)$ has a zero of order n at $x = 1$. Hence

$$\text{(B5)} \quad k(r_1) = \frac{(-)^{n+1}}{\alpha^{n-\frac{1}{2}}} \left(\frac{2}{\pi}\right)^{\frac{1}{2}} r_1^n \left(\frac{d}{r_1 dr_1}\right)^{n+1} \int_{r_1}^1 \frac{x \cosh\{\alpha(x^2- r_1^2)^{\frac{1}{2}}\}}{(x^2-r_1^2)^{\frac{1}{2}}}\ h(x)dx\ .$$

APPENDIX C

This Appendix is concerned with evaluating the behavior of f(w), p(w) and d(w) near w = 1. It is apparent from (47) that there are two different types of term to be considered for f, namely a typical member of the series and the member outside the series. There is a distinct similarity between (46) and (47) and we can deal with the first members of both at the same time by evaluating

$$2i\alpha w^{\frac{1}{2}(m-1)} + \frac{2\alpha e^{i\alpha(w-1)-\frac{1}{4}\pi i}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_0^\infty \frac{(1-it)^{\frac{1}{2}m} t^{\frac{1}{2}} e^{-\alpha t}}{1-w-it}\, dt$$

and then giving m the values 1 and 5 respectively for f and p.

Now

$$\int_0^\infty \frac{(1-it)^{\frac{1}{2}m} t^{\frac{1}{2}} e^{-\alpha t}}{1-w-it}\, dt = i\int_0^\infty \frac{(1-it)^{\frac{1}{2}m}}{t^{\frac{1}{2}}} e^{-\alpha t} dt - i(1-w)\int_0^\infty \frac{e^{-\alpha t}}{t^{\frac{1}{2}}(1-w-it)}\, dt$$

$$+ (1-w)\int_0^\infty \frac{K_m(t)}{t^{\frac{1}{2}}} e^{-\alpha t} dt + (1-w)^2 \int_0^\infty \frac{\frac{1}{2}mi e^{-\alpha t}}{t^{\frac{1}{2}}(1-w-it)}\, dt$$

$$- (1-w)^2 \int_0^\infty \frac{K_m(t) + \frac{1}{2}mi}{t^{\frac{1}{2}}(1-w-it)} e^{-\alpha t} dt$$

where $K_m(t) = \{(1-it)^{\frac{1}{2}m} - 1\}/t$. Also

$$\int_0^\infty \frac{e^{-\alpha t}}{t^{\frac{1}{2}}(1-w-it)}\,dt = i\pi^{\frac{1}{2}}e^{i\alpha(1-w)}\int_\alpha^\infty \frac{e^{-i(1-w)x}}{x^{\frac{1}{2}}}\,dx$$

$$= i\pi^{\frac{1}{2}}e^{i\alpha(1-w)}\left\{e^{-\frac{1}{4}\pi i}\pi^{\frac{1}{2}}(1-w)^{-\frac{1}{2}} - \int_0^{\alpha(1-w)} \frac{e^{-ix}}{(1-w)^{\frac{1}{2}}x^{\frac{1}{2}}}\,dx\right\}$$

$$= \pi e^{\frac{1}{4}\pi i+i\alpha(1-w)}\left[(1-w)^{-\frac{1}{2}} - \frac{e^{\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}}\left\{2\alpha^{\frac{1}{2}} - \frac{2i}{3}\alpha^{\frac{3}{2}}(1-w)\right\} + O\{(1-w)^2\}\right].$$

Hence

$$\int_0^\infty \frac{(1-it)^{\frac{1}{2}m}t^{\frac{1}{2}}e^{-\alpha t}}{1-w-it}\,dt = i\int_0^\infty \frac{(1-it)^{\frac{1}{2}m}}{t^{\frac{1}{2}}}\,e^{-\alpha t}dt + (1-w)\int_0^\infty \frac{K_m(t)}{t^{\frac{1}{2}}}\,e^{-\alpha t}dt$$

$$- i(1-w)^2\int_0^\infty \frac{K_m(t)+\frac{1}{2}mi}{t^{3/2}}\,e^{-\alpha t}dt$$

$$+\left\{\frac{1}{2}mi(1-w)^2 - i(1-w)\right\}\pi e^{\frac{1}{4}\pi i+i\alpha(1-w)}\left[(1-w)^{-\frac{1}{2}} - \frac{e^{\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}}\{2\alpha^{\frac{1}{2}} - \frac{2i}{3}\alpha^{\frac{3}{2}}(1-w)\}\right]$$

$$+ O\{(1-w)^{5/2}\}.$$

Therefore

$$2i\alpha w^{\frac{1}{2}(m-1)} + \frac{2\alpha e^{i\alpha(w-1)-\frac{1}{4}\pi i}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_{0}^{\infty} \frac{(1-it)^{\frac{1}{2}m} t^{\frac{1}{2}} e^{-\alpha t}}{1-w-it}\, dt$$

$$= \frac{2\alpha e^{-\frac{1}{4}\pi i}}{\pi(1-w)^{\frac{1}{2}}} \Bigg[\Bigg\{ 1 + (\tfrac{1}{2} - i\alpha)(1-w) + (\tfrac{3}{8} - \tfrac{1}{2} i\alpha - \tfrac{1}{2}\alpha^{2})(1-w)^{2} \Bigg\}$$

(C1)

$$\Bigg\{ i \int_{0}^{\infty} \frac{(1-it)^{\frac{1}{2}m}}{t^{\frac{1}{2}}} e^{-\alpha t} dt + (1-w) \int_{0}^{\infty} \frac{K_{m}(t)}{t^{\frac{1}{2}}} e^{-\alpha t} dt$$

$$- i(1-w)^{2} \int_{0}^{\infty} \frac{K_{m}(t) + \frac{1}{2}mi}{t^{3/2}} e^{-\alpha t} dt \Bigg\} - \pi^{\frac{1}{2}} \alpha^{\frac{1}{2}} (1-w) \Bigg\{ 2 - (\frac{2i\alpha}{3} + m - 1)(1-w) \Bigg\} \Bigg] .$$

Turn now to the term

$$\frac{H_{n}(w) e^{-i\alpha w}}{w} + \frac{e^{-i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_{1}^{\infty} (\frac{t-1}{t})^{\frac{1}{2}} \frac{H_{n}(t) e^{-2i\alpha t}}{t-w}\, dt .$$

We have

$$\int_{1}^{\infty} (\frac{t-1}{t})^{\frac{1}{2}} \frac{H_{n}(t) e^{-2i\alpha t}}{t-w}\, dt = \int_{1}^{\infty} \frac{H_{n}(t) e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}\, dt - (1-w) \int_{1}^{\infty} \frac{H_{n}(1) e^{-2i\alpha}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}(t-w)}\, dt$$

$$- (1-w) \int_{1}^{\infty} \frac{J_{n}(t)}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}\, dt + (1-w)^{2} \int_{1}^{\infty} \frac{J_{n}(1) dt}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}(t-w)}$$

$$+ (1-w)^{2} \int_{1}^{\infty} (\frac{t-1}{t})^{\frac{1}{2}} \frac{J_{n}(t) - J_{n}(1)}{(t-1)(t-w)}\, dt$$

where $J_n(t) = \{H_n(t)e^{-2i\alpha t} - H_n(1)e^{-2i\alpha}\}/(t-1)$. The substitution $t = (1-x^2)^{-1}$ shows that for $0 < w < 1$,

$$\int_1^\infty \frac{dt}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}(t-w)} = \frac{2}{w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \tan^{-1} \left(\frac{w}{1-w}\right)^{\frac{1}{2}}$$

$$= \frac{2}{w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \left[\frac{1}{2}\pi - (1-w)^{\frac{1}{2}}\{1 + \tfrac{1}{6}(1-w)\}\right] + O\{(1-w)^2\} .$$

Hence

$$\int_1^\infty \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \frac{H_n(t)e^{-2i\alpha t}}{t - w}\, dt = \int_1^\infty \frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}\, dt - (1-w)\int_1^\infty \frac{J_n(t)}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}\, dt$$

$$+ (1-w)^2 \int_1^\infty \frac{J_n(t)-J_n(1)}{t^{\frac{1}{2}}(t-1)^{3/2}}\, dt + \{J_n(1)(1-w)^2 - H_n(1)e^{-2i\alpha}(1-w)\}$$

$$\times \frac{2}{w^{\frac{1}{2}}} \left[\frac{1}{2}\pi\,(1-w)^{-\frac{1}{2}} - \{1 + \tfrac{1}{6}(1-w)\}\right] + O\{(1-w)^{5/2}\} .$$

Therefore

$$\frac{H_n(w)e^{-i\alpha w}}{w} + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} \int_1^\infty \left(\frac{t-1}{t}\right)^{\frac{1}{2}} \frac{H_n(t)e^{-2i\alpha t}}{t - w}\, dt = \frac{e^{i\alpha}}{\pi(1-w)^{\frac{1}{2}}} \left\{1 + (\tfrac{1}{2} - i\alpha)(1-w)\right.$$

$$\left. + (\tfrac{3}{8} - \tfrac{1}{2}i\alpha - \tfrac{1}{2}\alpha^2)(1-w)^2\right\} \left\{\int_1^\infty \frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}\, dt - (1-w)\int_1^\infty \frac{J_n(t)dt}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}\right.$$

$$\left. + (1-w)^2 \int_1^\infty \frac{J_n(t)-J_n(1)}{t^{\frac{1}{2}}(t-1)^{3/2}}\, dt + 2H_n(1)e^{-2i\alpha}(1-w) - 2J_n(1)(1-w)^2 + \tfrac{4}{3}H_n(1)e^{-2i\alpha}(1-w)^2\right\}$$

This formula can be somewhat simplified as follows:

$$\int_1^\infty \frac{J_n(t)}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}\,dt = \int_1^\infty \frac{H_n(t)e^{-2i\alpha t} - H_n(1)e^{-2i\alpha}}{t^{\frac{1}{2}}(t-1)^{3/2}}\,dt$$

$$= 2\int_1^\infty \frac{1}{(t-1)^{\frac{1}{2}}}\left[\left\{\frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}' + \frac{H_n(1)e^{-2i\alpha}}{2t^{3/2}}\right]dt$$

on integrating by parts, the factor $(t-1)^{-3/2}$ being integrated. The prime indicates a derivation with respect to t. Since

$$\int_1^\infty (t-1)^{-\frac{1}{2}}t^{-3/2}dt = 2$$

the last term of the integral is $2H_n(1)e^{-2i\alpha}$. Proceeding in the same way with the other integral containing $J_n(t)$ we find

$$\frac{H_n(w)e^{-i\alpha w}}{w} + \frac{e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}}\int_1^\infty \left(\frac{t-1}{t}\right)^{\frac{1}{2}}\frac{H_n(t)e^{-2i\alpha t}}{t - w}\,dt$$

$$\text{(C2)}\qquad = \frac{e^{i\alpha}}{\pi(1-w)^{\frac{1}{2}}}\left\{1 + \left(\frac{1}{2} - i\alpha\right)(1-w) + \left(\frac{3}{8} - \frac{1}{2}i\alpha - \frac{1}{2}\alpha^2\right)(1-w)^2\right\}\left[\int_1^\infty \frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}\,dt\right.$$

$$\left. - 2(1-w)\int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}}\left\{\frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}' + \frac{4}{3}(1-w)^2\int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}}\left\{\frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}''\right] + O\{(1-w)^2$$

With regard to d(w) the series term may be dealt with as in (C2). As for the first term

$$\frac{2e^{i\alpha w}}{\pi w^{\frac{1}{2}}(1-w)^{\frac{1}{2}}} = \frac{2e^{i\alpha}}{\pi(1-w)^{\frac{1}{2}}}\left\{1+(\tfrac{1}{2}-i\alpha)(1-w)+(\tfrac{3}{8}-\tfrac{1}{2}i\alpha-\tfrac{1}{2}\alpha^2)(1-w)^2\right\} + 0\left\{(1-w)^{5/2}\right\} . \tag{C3}$$

Using (C1) and (C2) we have

$$\lambda_1 = \frac{2\alpha e^{\frac{1}{4}\pi i}}{\pi}\int_0^\infty \frac{(1-it)^{\frac{1}{2}}}{t^{\frac{1}{2}}}e^{-\alpha t}dt + \sum_{n=1}\frac{e^{i\alpha}}{\pi}\int_1^\infty \frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}dt , \tag{C4}$$

$$\begin{aligned}\lambda_2 &= \frac{2\alpha}{\pi}e^{-\frac{1}{4}\pi i}\left[(\tfrac{1}{2}-i\alpha)i\int_0^\infty (1-it)^{\frac{1}{2}}\frac{e^{-\alpha t}}{t^{\frac{1}{2}}}dt + \int_0^\infty \frac{K_1(t)}{t^{\frac{1}{2}}}e^{-\alpha t}dt - 2\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}\right]\\ &+ \sum_{n=1}\frac{e^{i\alpha}}{\pi}\left[(\tfrac{1}{2}-i\alpha)\int_1^\infty \frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}dt - 2\int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}}\left\{\frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}'\right]\\ &= (\tfrac{1}{2}-i\alpha)\lambda_1 + \frac{2\alpha e^{-\frac{1}{4}\pi i}}{\pi}\left\{\int_0^\infty \frac{K_1(t)}{t^{\frac{1}{2}}}e^{-\alpha t}dt - 2\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}\right\}\\ &- \sum_{n=1}\frac{2e^{i\alpha}}{\pi}\int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}}\left\{\frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}' ,\end{aligned} \tag{C5}$$

$$\begin{aligned}\lambda_3 &= (\tfrac{1}{8}+\tfrac{1}{2}i\alpha+\tfrac{1}{2}\alpha^2)\lambda_1 + (\tfrac{1}{2}-i\alpha)\lambda_2 - \frac{2\alpha e^{\frac{1}{4}\pi i}}{\pi}\left\{\tfrac{4}{3}\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}} + i\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}} + \int_0^\infty \frac{\{K_1(t)+\frac{1}{2}i\}e^{-xt}}{t^{3/2}}dt\right\}\\ &+ \sum_{n=1}\frac{4e^{i\alpha}}{3\pi}\int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}}\left\{\frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}'' .\end{aligned} \tag{C6}$$

Similarly

$$\mu_1 = \frac{2e^{i\alpha}}{\pi} + \sum_{n=1} \frac{e^{i\alpha}}{\pi} \int_1^\infty \frac{E_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}\, dt\ , \tag{C7}$$

$$\mu_2 = (\tfrac{1}{2} - i\alpha)\mu_1 - \sum_{n=1} \frac{2e^{i\alpha}}{\pi} \int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}} \left\{\frac{E_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}'\ , \tag{C8}$$

$$\mu_3 = (\tfrac{1}{8} + \tfrac{1}{2}i\alpha + \tfrac{1}{2}\alpha^2)\mu_1 + (\tfrac{1}{2} - i\alpha)\mu_2 + \sum_{n=1} \frac{4e^{i\alpha}}{3\pi} \int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}} \left\{\frac{E_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}'' \tag{C9}$$

and

$$\nu_1 = \frac{2\alpha e^{\frac{1}{4}\pi i}}{\pi} \int_0^\infty \frac{(1-it)^{\frac{5}{2}}}{t^{\frac{1}{2}}} e^{-\alpha t} dt + \sum_{n=1} \frac{e^{i\alpha}}{\pi} \int_1^\infty \frac{S_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}\, dt\ , \tag{C10}$$

$$\nu_2 = (\tfrac{1}{2} - i\alpha)\nu_1 + \frac{2\alpha}{\pi} e^{-\frac{1}{4}\pi i} \left\{\int_0^\infty \frac{K_5(t)}{t^{\frac{1}{2}}} e^{-\alpha t} dt - 2\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}\right\} - \sum_{n=1} \frac{2e^{i\alpha}}{\pi} \int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}} \left\{\frac{S_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}'\ , \tag{C11}$$

$$\nu_3 = (\tfrac{1}{8} + \tfrac{1}{2}i\alpha + \tfrac{1}{2}\alpha^2)\nu_1 + (\tfrac{1}{2} - i\alpha)\nu_2 - \frac{2\alpha}{\pi} e^{\frac{1}{4}\pi i}\left\{\tfrac{4}{3}\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}} + 5i\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}} + \int_0^\infty \frac{K_5(t) + \frac{5}{2}i}{t^{3/2}} e^{-\alpha t} dt\right\} + \sum \frac{4e^{i\alpha}}{3\pi} \int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}} \left\{\frac{S_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}''\ . \tag{C12}$$

Appendix D

The asymptotic development of the coefficients $\lambda_1, \lambda_2, \ldots, \nu_3$ is a straightforward matter in principle but the detail is rather tedious. Both λ_1 and ν_1 can be dealt with at the same time by considering

$$\frac{2\alpha}{\pi} e^{\frac{1}{4}\pi i} \int_0^\infty \frac{(1-it)^{\frac{1}{2}m}}{t^{\frac{1}{2}}} e^{-\alpha t} dt$$

with m=1 and m=5 in the two cases. Now

$$\frac{2\alpha e^{\frac{1}{4}\pi i}}{\pi} \int_0^\infty \frac{(1-it)^{\frac{1}{2}m}}{t^{\frac{1}{2}}} e^{-\alpha t} dt = \frac{2\alpha}{\pi} e^{\frac{1}{4}\pi i} \int_0^\infty \frac{e^{-\alpha t}}{t^{\frac{1}{2}}} \left\{1 - \frac{1}{2}mit - \frac{1}{4}m(\frac{1}{2}m - 1)t^2 \right.$$

$$\left. + \frac{i}{12}m\,(\frac{1}{2}m - 1)(\frac{1}{2}m - 2)t^3\right\} dt + O(\alpha^{-\frac{7}{2}}) \tag{D1}$$

$$= \frac{2\alpha^{\frac{1}{2}} e^{\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}} \left\{1 - \frac{im}{4\alpha} - \frac{3m}{16\alpha^2}(\frac{1}{2}m - 1) + \frac{5im}{32\alpha^3}(\frac{1}{2}m - 1)(\frac{1}{2}m - 2)\right\} + O(\alpha^{-\frac{7}{2}}).$$

For the series in the coefficients we can use the result (Jones [17]), valid for $\alpha y \gg 1$,

$$H_n(y) \sim \frac{i}{\pi} \left(\frac{y}{1+y}\right)^{\frac{1}{2}} h_n(y) - \frac{1}{\pi\alpha y^{\frac{1}{2}}} \left\{(e^{2i\alpha y} - i)h_{n,o} + \frac{9h_n(0)}{128\alpha y}(e^{2i\alpha y} + i)\right\} \tag{D2}$$

where

$$h_{n,o} = \frac{i}{8}\left[h_n(0) + \frac{9i}{16\alpha}\left\{\frac{1}{2} h_n(0) - h_n'(0)\right\}\right]. \tag{D3}$$

From this we see that

$$\frac{e^{i\alpha}}{\pi}\int_1^\infty \frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}\,dt = \frac{ie^{i\alpha}}{\pi^2}\int_1^\infty \frac{e^{-2i\alpha t}}{(t-1)^{\frac{1}{2}}}\left\{\frac{h_n(t)}{(1+t)^{\frac{1}{2}}} + \frac{h_{n,o}}{\alpha t} - \frac{9h_n(0)}{128\alpha^2t^2}\right\}dt$$

$$- \frac{e^{i\alpha}}{\pi^2\alpha}\int_1^\infty \frac{dt}{t(t-1)^{\frac{1}{2}}}\left\{h_{n,o} + \frac{9h_n(0)}{128\alpha t}\right\} + O\left(\frac{h_n}{\alpha^3}\right)$$

$$= \frac{e^{\frac{1}{4}\pi i - i\alpha}}{\pi^2}\int_0^\infty \frac{e^{-2\alpha t}}{t^{\frac{1}{2}}}\left\{\frac{h_n(1-it)}{(2-it)^{\frac{1}{2}}} + \frac{h_{n,o}}{\alpha(1-it)} - \frac{9h_n(0)}{128\alpha^2(1-it)^2}\right\}dt$$

$$- \frac{e^{i\alpha}}{\pi\alpha}\left\{h_{n,o} + \frac{9h_n(0)}{256\alpha}\right\} + O\left(\frac{h_n}{\alpha^3}\right)$$

on deforming the contour of the first integral into the line joining 1 to 1 - i∞ and using

$$\int_1^\infty \frac{dt}{t(t-1)^{\frac{1}{2}}} = \pi, \qquad \int_1^\infty \frac{dt}{t^2(t-1)^{\frac{1}{2}}} = \frac{1}{2}\pi\ .$$

Therefore

$$\frac{e^{i\alpha}}{\pi}\int_1^\infty \frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}}\,dt = \frac{e^{\frac{1}{4}\pi i-i\alpha}}{\pi^2}\int_0^\infty \frac{e^{-2\alpha t}}{t^{\frac{1}{2}}}\left[\frac{h_n(1)}{2^{\frac{1}{2}}} + \frac{it}{2^{\frac{1}{2}}}\left\{\frac{1}{4}h_n(1) - h_n'(1)\right\}\right.$$

$$- \frac{t^2}{2^{\frac{3}{2}}}\left\{h_n''(1) - \frac{1}{2}h_n'(1) + \frac{3}{16}h_n(1)\right\} + \frac{h_{n,o}}{\alpha}(1+it)$$

$$\left.- \frac{9h_n(0)}{128\alpha^2}\right]dt - \frac{e^{i\alpha}}{\pi\alpha}\left\{h_{n,o} + \frac{9h_n(0)}{256\alpha}\right\} + O\left(\frac{h_n}{\alpha^3}\right)$$

(D4)

$$= \frac{e^{\frac{1}{4}\pi i-i\alpha}}{2\pi^{\frac{3}{2}}\alpha^{\frac{1}{2}}}\left[h_n(1) + \frac{i}{4\alpha}\left\{\frac{1}{4}h_n(1) - h_n'(1)\right\} - \frac{3}{32\alpha^2}\left\{h_n''(1)\right.\right.$$

$$\left.- \frac{1}{2}h_n'(1) + \frac{3}{16}h_n(1)\right\} + \frac{2^{\frac{1}{2}}h_{n,o}}{\alpha}\left(1 + \frac{i}{4\alpha}\right) - \left.\frac{9.2^{\frac{1}{2}}h_n(0)}{128\alpha^2}\right]$$

$$- \frac{e^{i\alpha}}{\pi\alpha}\left\{h_{n,o} + \frac{9h_n(0)}{256\alpha}\right\} + O\left(\frac{h_n}{\alpha^3}\right) .$$

To evaluate the various terms in (D4) we need

$$h_1(v) = \frac{\pi^{\frac{1}{2}}e^{-i\alpha-\frac{1}{4}\pi i}}{\alpha^{\frac{1}{2}}(1+v)}\left[1 + \frac{3i}{2\alpha}\left(\frac{1}{1+v} - \frac{1}{2}\right) + \frac{15}{4\alpha^2}\left\{\frac{1}{8} + \frac{1}{2(1+v)} - \frac{1}{(1+v)^2}\right\}\right] + O(\alpha^{-\frac{7}{2}}) .$$

Hence

$$h_1(1) = \frac{\pi^{\frac{1}{2}} e^{-i\alpha-\frac{1}{4}\pi i}}{2\alpha^{\frac{1}{2}}} \left(1 + \frac{15}{32\alpha^2}\right) + O(\alpha^{-\frac{7}{2}}) , \tag{D5}$$

$$h_1(0) = \frac{\pi^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i} \left(1 + \frac{3i}{4\alpha} - \frac{45}{32\alpha^2}\right) + O(\alpha^{-\frac{7}{2}}) , \tag{D6}$$

$$h_1'(1) = - \frac{\pi^{\frac{1}{2}}}{4\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i} \left(1 + \frac{3i}{4\alpha}\right) \tag{D7}$$

$$h_1'(0) = - \frac{\pi^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i} + O(\alpha^{-\frac{3}{2}}) , \tag{D8}$$

$$h_1''(1) = \frac{\pi^{\frac{1}{2}}}{4\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i} + O(\alpha^{-\frac{3}{2}}) , \tag{D9}$$

$$h_{1,o} = \frac{i\pi^{\frac{1}{2}} e^{-i\alpha-\frac{1}{4}\pi i}}{8\alpha^{\frac{1}{2}}} \left(1 + \frac{51i}{32\alpha}\right) + O(\alpha^{-\frac{5}{2}}) . \tag{D10}$$

In our earlier paper it is shown that, for $n \geq 2$,

$$h_n(1) = \frac{h_{n-1,o}}{\alpha} (1-2^{\frac{1}{2}}) + \frac{e^{-2i\alpha-\frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}} h_{n-1}(1) - \frac{9h_{n-1}(0)}{128\alpha^2} \left(\frac{3}{2} - 2^{\frac{1}{2}}\right) + O\left(\frac{h_{n-1}}{\alpha^{\frac{5}{2}}}\right) , \tag{D11}$$

$$h_n'(1) = \frac{h_{n-1,o}}{\alpha} \left(\frac{3}{2^{\frac{3}{2}}} - 1\right) - \frac{e^{-2i\alpha-\frac{1}{4}\pi i}}{32\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}} h_{n-1}(1) - \frac{9h_{n-1}(0)}{256\alpha^2} \left(\frac{7}{2^{\frac{1}{2}}} - 5\right) + O\left(\frac{h_{n-1}}{\alpha^{\frac{5}{2}}}\right) . \tag{D12}$$

$$\text{(D13)}\quad h_n(0) = -\frac{1}{2\alpha} h_{n-1,o} + \frac{e^{-2i\alpha-\frac{1}{4}\pi i}}{8\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}} h_{n-1}(1) + \frac{9h_{n-1}(0)}{1024\alpha^2} + 0\left(\frac{h_{n-1}}{\alpha^{\frac{5}{2}}}\right) ,$$

$$\text{(D14)}\quad h_n'(0) = \frac{h_{n-1,o}}{8\alpha} - \frac{e^{-2i\alpha-\frac{1}{4}\pi i}}{8\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}} h_{n-1}(1) + \frac{9h_{n-1}(0)}{2048\alpha^2} + 0\left(\frac{h_{n-1}}{\alpha^{\frac{5}{2}}}\right) ,$$

$$\text{(D15)}\quad h_{n,o} = -\frac{i}{16\alpha} h_{n-1,o} + \frac{e^{\frac{1}{4}\pi i-2i\alpha}}{64\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}} h_{n-1}(1) + 0(\frac{h_{n-1}}{\alpha^2}) .$$

Since $e_1(v) = 2$ we have

$$\text{(D16)}\, e_1(1) = 2,\quad e_1(0) = 2,\quad e_1'(1) = 0,\quad e_1'(0) = 0,\quad e_1''(1) = 0,\quad e_{1,o} = \frac{i}{4}(1 + \frac{9i}{32\alpha}) .$$

Formulae (D11) - (D15) are also valid with h_n replaced by e_n.

The corresponding results for s_n are

$$\text{(D17)}\qquad s_1(1) = \frac{\pi^{\frac{1}{2}}}{2\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i}(1 - \frac{3i}{\alpha} - \frac{105}{32\alpha^2}) + 0(\alpha^{-\frac{7}{2}}) ,$$

$$\text{(D18)}\qquad s_1(0) = \frac{\pi^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i}(1- \frac{9i}{4\alpha}) + 0(\alpha^{-\frac{5}{2}}) ,$$

$$\text{(D19)}\qquad s_1'(1) = -\frac{\pi^{\frac{1}{2}}}{4\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i}(1 - \frac{9i}{4\alpha}) + 0(\alpha^{-\frac{5}{2}}) ,$$

(D20) $$s_1'(0) = -\frac{\pi^{\frac{1}{2}}}{\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i} + O(\alpha^{-\frac{3}{2}}) ,$$

(D21) $$s_1''(1) = -\frac{\pi^{\frac{1}{2}}}{4\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i} + O(\alpha^{-\frac{3}{2}}) ,$$

(D22) $$s_{1,o} = \frac{i\pi^{\frac{1}{2}}}{8\alpha^{\frac{1}{2}}} e^{-i\alpha-\frac{1}{4}\pi i}\left(1 - \frac{45i}{32\alpha}\right) + O(\alpha^{-\frac{5}{2}}) .$$

The formulae (D11) - (D15) hold with s_n for h_n.

It follows from (C4), (D1), (D4)-(D15) that

(D23)
$$\lambda_1 = \frac{2\alpha^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} e^{\frac{1}{4}\pi i}\left(1 - \frac{i}{4\alpha} + \frac{1}{32\alpha^2} + \frac{5i}{128\alpha^3}\right) + \frac{e^{-2i\alpha}}{4\pi\alpha}\left(1 + \frac{7i}{16\alpha} + \frac{e^{-2i\alpha-\frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{3/2}}\right) + O(\alpha^{-3}) .$$

From (C7), (D4), (D16) and (D11)-(D15) we have

(D24) $$\mu_1 = \frac{2e^{i\alpha}}{\pi}\left(1 - \frac{i}{8\alpha} - \frac{1}{128\alpha^2}\right) + \frac{e^{\frac{1}{4}\pi i-i\alpha}}{\pi^{\frac{3}{2}}\alpha^{\frac{1}{2}}}\left(1 + \frac{3i}{16\alpha} - \frac{113}{512\alpha^2} + \frac{e^{-2i\alpha-\frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\right) + O(\alpha^{-3}) .$$

From (C10), (D1), (D17)-(D22) results

(D25) $$\nu_1 = \frac{2\alpha^{\frac{1}{2}}}{\pi^{\frac{1}{2}}} e^{\frac{1}{4}\pi i}\left(1 - \frac{5i}{4\alpha} - \frac{47}{32\alpha^2} + \frac{89i}{128\alpha^3}\right) + \frac{e^{-2i\alpha}}{4\pi\alpha}\left(1 - \frac{41i}{16\alpha} + \frac{e^{-2i\alpha-\frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\right) + O(\alpha^{-3}) .$$

Proceed in a similar manner for the coefficient λ_2, μ_2 and ν_2. We have

(D26)

$$\int_0^\infty \frac{K_m(t)}{t^{\frac{1}{2}}} e^{-\alpha t} dt - 2\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}} = \int_0^\infty \frac{e^{-\alpha t}}{t^{\frac{1}{2}}} \left\{ -\frac{1}{2}mi - \frac{1}{4}m(\frac{1}{2}m-1)t + \frac{i}{12}m\,(\frac{1}{2}m-1)(\frac{1}{2}m-2)t^2 \right\} dt$$

$$- 2\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}} + 0(\alpha^{-\frac{7}{2}})$$

$$= -2\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}} \left\{ 1 + \frac{im}{4\alpha} + \frac{m}{16\alpha^2}(\frac{1}{2}m-1) - \frac{im}{32\alpha^3}(\frac{1}{2}m-1)(\frac{1}{2}m-2) \right\} + 0(\alpha^{-\frac{7}{2}}) .$$

Also

$$\frac{2e^{i\alpha}}{\pi} \int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}} \left\{ \frac{H_n(t) e^{-2i\alpha t}}{t^{\frac{1}{2}}} \right\}' = - \frac{4i\alpha e^{i\alpha}}{\pi} \int_1^\infty \frac{H_n(t) e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}} dt$$

$$+ \frac{2e^{i\alpha}}{\pi} \int_1^\infty \left\{ \frac{H_n(t)}{t^{\frac{1}{2}}} \right\}' \frac{e^{-2i\alpha t}}{(t-1)^{\frac{1}{2}}} dt$$

and

$$\frac{2e^{i\alpha}}{\pi} \int_1^\infty \left\{ \frac{H_n(t)}{t^{\frac{1}{2}}} \right\}' \frac{e^{-2i\alpha t}}{(t-1)^{\frac{1}{2}}} dt = \frac{2ie^{i\alpha}}{\pi^2} \int_1^\infty \frac{e^{-2i\alpha t}}{(t-1)^{\frac{1}{2}}} \left\{ \frac{h_n'(t)}{(1+t)^{\frac{1}{2}}} - \frac{h_n(t)}{2(1+t)^{\frac{3}{2}}} - \frac{h_{n,o}}{\alpha t^2} \right\} dt$$

$$- \frac{4ie^{i\alpha}}{\pi^2} \int_1^\infty \frac{dt}{t(t-1)^{\frac{1}{2}}} \left\{ h_{n,o} + \frac{9h_n(0)}{128\alpha t} \right\} + \frac{2e^{i\alpha}}{\pi^2\alpha} \int_1^\infty \frac{h_{n,o} dt}{t^2(t-1)^{\frac{1}{2}}} + 0(\frac{h_n}{\alpha^2})$$

$$= \frac{2e^{\frac{1}{4}\pi i - i\alpha}}{\pi^{\frac{3}{2}}\alpha^{\frac{1}{2}}} \left[\frac{1}{2}\left\{h_n'(1) - \frac{1}{4}h_n(1)\right\} - \frac{i}{32\alpha}\left\{\frac{3}{4}h_n(1) - 2h_n'(1) + 4h_n''(1)\right\}\right.$$

(D28)

$$\left. - \frac{1}{2^{\frac{1}{2}}\alpha} h_{n,o}\right] - \frac{4ie^{i\alpha}}{\pi}\left\{h_{n,o} + \frac{9h_n(0)}{256\alpha} + \frac{ih_{n,o}}{4\alpha}\right\} + O\left(\frac{h_n}{\alpha^2}\right)$$

in the same way as was employed in deriving (D4). Use (D4) for the first term in (D27) and then

$$\frac{2e^{i\alpha}}{\pi}\int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}}\left\{\frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}' = -\frac{2i\alpha^{\frac{1}{2}}}{\pi^{\frac{3}{2}}} e^{\frac{1}{4}\pi i - i\alpha}\left[h_n(1) + \frac{i}{4\alpha}\left\{h_n'(1) - \frac{1}{4}h_n(1)\right\}\right.$$

(D29)

$$\left. + \frac{1}{32\alpha^2}\left\{h_n''(1) - \frac{1}{2}h_n'(1) + \frac{3}{16}h_n(1)\right\} + \frac{2^{\frac{1}{2}}h_{n,o}}{\alpha}\left(1 - \frac{i}{4\alpha}\right) - \frac{9.2^{\frac{1}{2}}h_n(0)}{128\alpha^2}\right] + \frac{h_{n,o}e^{i\alpha}}{\pi\alpha} + O\left(\frac{h_n}{\alpha^2}\right).$$

The combination of (C5), (D26), (D29), (D5)-(D15) gives

(D30)

$$\lambda_2 = \left(\frac{1}{2} - i\alpha\right)\lambda_1 - \frac{4\alpha^{\frac{3}{2}}}{\pi^{\frac{1}{2}}} e^{-\frac{1}{4}\pi i}\left(1 + \frac{i}{4\alpha} - \frac{1}{32\alpha^2} + \frac{i}{128\alpha^3}\right) + \frac{i}{\pi} e^{-2i\alpha}\left(1 + \frac{i}{16\alpha} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\right) + O(\alpha^{-2}).$$

From (C8), (D29), (D16), (D11)-(D15) follows

$$\text{(D31)}\ \mu_2 = \left(\frac{1}{2} - i\alpha\right)\mu_1 - \frac{ie^{i\alpha}}{4\pi\alpha} + \frac{4i\alpha^{\frac{1}{2}}}{\pi^{\frac{3}{2}}} e^{\frac{1}{4}\pi i - i\alpha}\left(1 + \frac{i}{16\alpha} - \frac{45}{512\alpha^2} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\right) + O(\alpha^{-2}).$$

In addition, (C11), (D26), (D29), (D17)-(D22) give

(D32)
$$\nu_2 = (\tfrac{1}{2} - i\alpha)\nu_1 - \frac{4\alpha^{\frac{3}{2}} e^{-\frac{1}{4}\pi i}}{\pi^{\frac{1}{2}}} \left(1 + \frac{5i}{4\alpha} + \frac{15}{32\alpha^2} - \frac{11i}{128\alpha^3}\right) + \frac{ie^{-2i\alpha}}{\pi}\left(1 - \frac{47i}{16\alpha} + \frac{e^{-2i\alpha-\frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\right) + O(\alpha^{-2}).$$

The coefficients λ_3, μ_3 and ν_3 can be determined in a similar manner. Thus

$$\frac{4}{3}\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}} + im\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}} + \int_0^\infty \frac{K_m(t) + \frac{1}{2}mi}{t^{\frac{3}{2}}} e^{-\alpha t} dt = \frac{4}{3}\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}} + im\pi^{\frac{1}{2}}\alpha^{\frac{1}{2}}$$

(D33)
$$+ \int_0^\infty \frac{e^{-\alpha t}}{t^{\frac{1}{2}}} \left\{-\tfrac{1}{4}m(\tfrac{1}{2}m-1) + \frac{im}{12}(\tfrac{1}{2}m-1)(\tfrac{1}{2}m-2)\right\} dt + O(\alpha^{-\frac{5}{2}})$$

$$= \frac{4}{3}\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}} \left\{1 + \frac{3im}{4\alpha} - \frac{3m}{16\alpha^2}(\tfrac{1}{2}m-1) + \frac{im}{32\alpha^3}(\tfrac{1}{2}m-1)(\tfrac{1}{2}m-2)\right\} + O(\alpha^{-\frac{5}{2}}) .$$

Also

$$\frac{4e^{i\alpha}}{3\pi} \int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}} \left\{\frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}'' = - \frac{16a^2}{3\pi} e^{i\alpha} \int_1^\infty \frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}(t-1)^{\frac{1}{2}}} dt$$

$$- \frac{16i\alpha}{3\pi} e^{i\alpha} \int_1^\infty \frac{e^{-2i\alpha t}}{(t-1)^{\frac{1}{2}}} \left\{\frac{H_n(t)}{t^{\frac{1}{2}}}\right\}' dt + \frac{4e^{i\alpha}}{3\pi} \int_1^\infty \frac{e^{-2i\alpha t}}{(t-1)^{\frac{1}{2}}} \left\{\frac{H_n(t)}{t^{\frac{1}{2}}}\right\}'' dt .$$

The first integral is given by (D4) and the second by (D28). As for the third

$$\frac{4e^{i\alpha}}{3\pi}\int_1^\infty \frac{e^{-2i\alpha t}}{(t-1)^{\frac{1}{2}}}\left\{\frac{H_n(t)}{t^{\frac{1}{2}}}\right\}'' dt = \frac{4ie^{i\alpha}}{3\pi^2}\int_1^\infty \frac{e^{-2i\alpha t}}{(t-1)^{\frac{1}{2}}}\left\{\frac{h_n''(t)}{(1+t)^{\frac{1}{2}}} - \frac{h_n'(t)}{(1+t)^{\frac{3}{2}}} + \frac{3h_n(t)}{4(1+t)^{\frac{5}{2}}}\right\} dt$$

$$+ \frac{16\alpha e^{i\alpha}}{3\pi^2}\int_1^\infty \frac{dt}{t(t-1)^{\frac{1}{2}}}\left\{h_{n,o} + \frac{9h_n(0)}{128\alpha t}\right\} + \frac{16ie^{i\alpha}}{3\pi^2}\int_1^\infty \frac{h_{n,o}dt}{t^2(t-1)^{\frac{1}{2}}} + O(\frac{h_n}{\alpha})$$

$$= \frac{2e^{\frac{1}{4}\pi i-i\alpha}}{3\pi^{\frac{3}{2}}\alpha^{\frac{1}{2}}}\left\{h_n''(1) - \frac{1}{2}h_n'(1) + \frac{3}{16}h_n(1)\right\} + \frac{16\alpha e^{i\alpha}}{3\pi}\left\{h_{n,o} + \frac{9h_n(0)}{256\alpha} + \frac{ih_{n,o}}{2\alpha}\right\} + O(\frac{h_n}{\alpha}) .$$

Consequently

$$\frac{4e^{i\alpha}}{3\pi}\int_1^\infty \frac{dt}{(t-1)^{\frac{1}{2}}}\left\{\frac{H_n(t)e^{-2i\alpha t}}{t^{\frac{1}{2}}}\right\}'' = -\frac{8\alpha^{\frac{3}{2}}}{3\pi^{\frac{3}{2}}}e^{\frac{1}{4}\pi i-i\alpha}\Bigg[h_n(1) + \frac{3i}{4\alpha}\left\{h_n'(1) - \frac{1}{4}h_n(1)\right\}$$

(D34)

$$- \frac{3}{32\alpha^2}\left\{h_n''(1) - \frac{1}{2}h_n'(1) + \frac{3}{16}h_n(1)\right\} + \frac{2^{\frac{1}{2}}}{\alpha}h_{n,o}(1 - \frac{3i}{4\alpha}) - \frac{9.2^{\frac{1}{2}}}{128\alpha^2}h_n(0)\Bigg] + O(\frac{h_n}{\alpha})$$

By means of (C6), (D33), (D34), (D5)-(D15) we find

$$\lambda_3 = (\frac{1}{8} + \frac{1}{2}i\alpha + \frac{1}{2}\alpha^2)\lambda_1 + (\frac{1}{2} - i\alpha)\lambda_2 - \frac{8\alpha^{\frac{5}{2}}}{3\pi^{\frac{1}{2}}}e^{\frac{1}{4}\pi i}(1 + \frac{3i}{4\alpha} + \frac{3}{32\alpha^2} + \frac{3i}{128\alpha^3})$$

(D35)

$$- \frac{4\alpha}{3\pi}e^{-2i\alpha}(1 - \frac{5i}{16\alpha} + \frac{e^{-2i\alpha-\frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}) + O(\alpha^{-1}) .$$

A consequence of (C9), (D34), (D16), (D11)-(D15) is

$$\mu_3 = (\tfrac{1}{8} + \tfrac{1}{2}i\alpha + \tfrac{1}{2}\alpha^2)\mu_1 + (\tfrac{1}{2} - i\alpha)\mu_2 - \frac{16\alpha^{\frac{3}{2}}}{3\pi^{\frac{3}{2}}} e^{\frac{1}{4}\pi i - i\alpha}\left(1 - \frac{i}{16\alpha} - \frac{17}{512\alpha^2} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{3/2}}\right) + O(\alpha^{-1}) \tag{D36}$$

From (C12), (D33), (D34), (D17)-(D22) it follows that

$$\nu_3 = (\tfrac{1}{8} + \tfrac{1}{2}i\alpha + \tfrac{1}{2}\alpha^2)\nu_1 + (\tfrac{1}{2} - i\alpha)\nu_2 - \frac{8\alpha^{\frac{5}{2}}}{3\pi^{\frac{1}{2}}} e^{\frac{1}{4}\pi i}\left(1 + \frac{15i}{4\alpha} - \frac{45}{32\alpha^2} + \frac{15i}{128\alpha^3}\right)$$

(D37)

$$- \frac{4\alpha}{3\pi} e^{-2i\alpha}\left(1 - \frac{53i}{16\alpha} + \frac{e^{-2i\alpha - \frac{1}{4}\pi i}}{16\pi^{\frac{1}{2}}\alpha^{\frac{3}{2}}}\right) + O(\alpha^{-1}).$$

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